The arithmetic Hodge index theorem for adelic line bundles II: finitely generated fields

Xinyi Yuan, Shou-Wu Zhang

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1 Introduction

In the arithmetic intersection theory of Gillet–Soulé [GS1], for a projective variety X over a number field K with an integral model $\mathcal{X}$ over O_K and a Hermitian line bundle $\overline{\mathcal{L}}$ over $\mathcal{X}$, the height of a point $P \in X(K)$ is expressed as the arithmetic degree $\widehat{\deg}(\bar{P}^*\overline{\mathcal{L}})$ of $\overline{\mathcal{L}}$ at the corresponding arithmetic curve $\bar{P} \in \mathcal{X}(O_K)$. Thus many powerful tools from algebraic geometry and complex geometry can be used to study questions about heights, for example, in the proofs of the Mordell conjecture and the Bogomolov conjecture. In [Mo2, Mo3], Moriwaki developed a height theory for varieties over any finitely generated field K over $\mathbb{Q}$ of transcendental degree d, and proved a new case of the Bogomolov conjecture. In his theory, the height is the intersection number $\bar{P}^*\overline{\mathcal{L}} \cdot \overline{\mathcal{H}}_1 \cdots \overline{\mathcal{H}}_{d-1}$, where $\bar{P} \in \mathcal{X}(\mathcal{B})$ extends $P \in X(K)$, $\mathcal{X}/\mathcal{B}$ is a fixed projective and flat model of X/K over $\mathbb{Z}$, and $\overline{\mathcal{H}}_1, \cdots, \overline{\mathcal{H}}_{d-1}$ are fixed polarizations on B. One important feature of this theory is to allow people to prove theorems over $\mathbb{C}$ (or any field of characteristic 0) by arithmetic methods, since any variety over $\mathbb{C}$ is the base change of a variety over a finitely generated field. In fact, Moriwaki recovered Raynaud's theorem (over $\mathbb{C}$) on the Manin–Mumford conjecture from his Bogomolov conjecture.

Our goal of this paper is to develop a (slightly new) theory of adelic line bundles and *vector-valued heights* on projective varieties over finitely generated fields, and then use it to extend the following two results of [YZ] from number fields to finitely generated fields:

(1) the arithmetic Hodge index theorem for adelic line bundles on a projective variety over a finitely generated field K,

(2) a rigidity result of the sets of preperiodic points of polarizable endomorphisms of a projective variety over any field K.

One reason to use our new theory of adelic line bundles and *vector-valued heights* rather than the Moriwaki's polarization is that statements about the equality part of the Hodge index theorem in Moriwaki's setting would be too weak for applications in (1) and (2). This was already the case in Kawaguchi's work [Ka] for families of curves.

The exposition of this paper uses a combination of Arakelov theory (cf. [Ar, GS1]) and a slight generalization of Berkovich analytic spaces (cf. [Be]). In the following, we introduce our main results with precise definitions.

1.1 Arithmetic Hodge index theorem

In [YZ], we have proved an arithmetic Hodge index theorem for adelic line
bundles over a projective variety over a number field. It extends the result
of Faltings [Fal], Hriljac [Hr] and Moriwaki [Mo1]. Here we describe the
generalization to finitely generated fields.

Our absolute base scheme for the integral models will be either $\operatorname{Spec}\mathbb{Z}$
or $\operatorname{Spec} k$ for a finite field k. We refer them as *the arithmetic case* and *the
geometric case* respectively. By an *open variety*, we mean an integral scheme
which is flat and quasi-projective over $\mathbb{Z}$ in the arithmetic case, or a quasi-
projective variety over k in the geometric case.

Let $\pi : \mathcal{U} \to \mathcal{V}$ be a projective and flat morphism of open varieties in either
the arithmetic case or the geometric case. We construct a group $\widehat{\operatorname{Pic}}(\mathcal{U})_{\mathrm{int}}$ of
integrable metrized line bundles on $\mathcal{U}$ as follows. Define *the group of model
metrized line bundles* by

$$\widehat{\operatorname{Pic}}(\mathcal{U})_{\mathrm{mod}} := \varinjlim_{\mathcal{X} \to \mathcal{B}} \widehat{\operatorname{Pic}}(\mathcal{X})_{\mathbb{Q}},$$

where the limit runs over all *projective and flat models* $\mathcal{X} \to \mathcal{B}$ of $\mathcal{U} \to \mathcal{V}$.
Namely, $\mathcal{X}$ and $\mathcal{B}$ are integral schemes, projective and flat over either $\mathbb{Z}$ or k
and containing $\mathcal{U}$ and $\mathcal{V}$ as open subschemes, such that the morphism $\mathcal{X} \to \mathcal{B}$
extends $\mathcal{U} \to \mathcal{V}$. And $\widehat{\operatorname{Pic}}(\mathcal{X})$ denotes the group of Hermitian line bundles in
the arithmetic case or the usual Picard group in the geometry case.

Define a topology on $\widehat{\operatorname{Pic}}(\mathcal{U})_{\mathrm{mod}}$ using a strictly effective divisor $\widehat{D}$ with
support $|D| = \mathcal{X} \setminus \mathcal{U}$. The completion is denoted by $\widehat{\operatorname{Pic}}(\mathcal{U})_{\mathrm{cont}}$. More
precisely, an element $\overline{\mathcal{L}}$ in this group is represented by data $(\overline{\mathcal{L}}_i, \ell_{i,j})$ ($i \geq
j \geq 0$) with a convergence condition, where

(1) $\overline{\mathcal{L}}_i$ is a sequence of line bundles on models $\mathcal{X}_i$ with compatible morphisms
$\pi_{i,j} : \mathcal{X}_i \to \mathcal{X}_j$ ($i \geq j \geq 0$) of $\mathcal{U}$-models, and

(2) $\ell_{i,j}$ is a compatible system of rational sections of $\mathcal{L}_i \otimes \pi_{i,j}^* \mathcal{L}_j^{-1}$ whose zero
loci are supported on $\mathcal{X}_i \setminus \mathcal{X}_{i,\mathcal{U}}$.

The convergence condition is as follows. For any $\epsilon > 0$, there is an i_0 such
that for any $i \geq j \geq i_0$, the divisors

$$\epsilon \pi_{i,0}^* \widehat{D} \pm \widehat{\operatorname{div}}(\ell_{i,j})$$

are both strictly effective.

An element of $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}}$ is called *nef* it is equal to the limit of a sequence of nef Hermitian line bundles. An element of $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}}$ is called *integrable* if it is equal to the difference of two nef adelic line bundles in $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}}$. Denote by $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{nef}}$ the cone of nef adelic line bundles on X, and by $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}}$ the group of integrable adelic line bundles on X.

Now we consider the generic fiber. Let K be a finitely generated field (over a prime field). It is of one of the following two types:

- arithmetic case: $\mathrm{char}(K) = 0$, and thus K is finitely generated over $\mathbb{Q}$ of transcendental degree d,

- geometric case: $\mathrm{char}(K) > 0$, and thus K is finitely generated over a finite field k of transcendental degree $d + 1$.

By an *open model* of K, we mean an open variety (in either case) with function field K.

Let $\pi : X \to \mathrm{Spec}\, K$ be a projective variety. Let $\mathcal{U} \to \mathcal{V}$ be an open model of π. Namely, $\mathcal{V}$ is an open model of K, and $\mathcal{U} \to \mathcal{V}$ is a projective and flat morphism of open varieties with generic fiber π. Then we define

$$\widehat{\mathrm{Pic}}(X)_{\mathrm{int}} = \varinjlim_{\mathcal{U} \to \mathcal{V}} \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}}.$$

In particular, when $X = \mathrm{Spec}\, K$ and π is the identity map, we get

$$\widehat{\mathrm{Pic}}(K)_{\mathrm{int}} = \varinjlim_{\mathcal{V}} \widehat{\mathrm{Pic}}(\mathcal{V})_{\mathrm{int}}.$$

The groups $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ and $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ can be realized as metrized line bundles in Berkovich spaces X^{an} and $(\mathrm{Spec}\, K)^{\mathrm{an}}$.

If K is a number field, the group $\widehat{\mathrm{Pic}}(X)_{\mathrm{cont}}$ recovers the group $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ of $\mathbb{Q}$-linear combinations of adelic line bundles on X introduced in [Zh2]. Furthermore, in this case, $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ is equal to $\widehat{\mathrm{Pic}}(K)$, since any adelic line bundle over a number field is integrable.

Go back to the general X and K. By the limit process, we have natural pull-back maps

$$\widehat{\mathrm{Pic}}(K')_{\mathrm{int}} \longrightarrow \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}, \quad \widehat{\mathrm{Pic}}(K)_{\mathrm{int}} \xrightarrow{\pi^*} \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}.$$

Here K' is any subfield of K. In the other direction, we can define an (absolute) intersection product

$$\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}^{d+1} \longrightarrow \mathbb{R}, \qquad (\overline{H}_1, \cdots, \overline{H}_{d+1}) \longmapsto \overline{H}_1 \cdots \overline{H}_{d+1},$$

and a relative intersection product

$$\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}, \qquad (\overline{L}_1, \cdots, \overline{L}_{n+1}) \longmapsto \pi_*(\overline{L}_1 \cdots \overline{L}_{n+1}).$$

By the limit process, we have the cones $\widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$ and $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$ of nef adelic line bundles. We introduce the following further positivity notions.

Definition 1.1. *Let* $\overline{L}, \overline{M} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ *and* $\overline{H} \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$. *We define the following notions.*

(1) $\overline{H} \geq 0$ *if* $\overline{H}$ *is* pseudo-effective, *i.e., the top intersection number* $\overline{H} \cdot \overline{N}_1 \cdots \overline{N}_d \geq 0$ *for any* $\overline{N}_1, \cdots, \overline{N}_d$ *in* $\widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$.

(2) $\overline{H} \equiv 0$ *if* $\overline{H}$ *is* numerically trivial, *i.e., the top intersection number* $\overline{H} \cdot \overline{N}_1 \cdots \overline{N}_d = 0$ *for any* $\overline{N}_1, \cdots, \overline{N}_d$ *in* $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$.

(3) $\overline{L} \gg 0$ *if* L *is ample, and* $\overline{L} - \overline{N}$ *is nef for some* $\overline{N} \in \widehat{\mathrm{Pic}}(k_1)$ *with* $\widehat{\deg}(\overline{N}) > 0$ *and for some subfield* k_1 *of* K. *Here* k_1 *is* $\mathbb{Q}$ *in the arithmetic case, and a finitely generated extension of* k *of transcendental degree 1 in the geometric case. The adelic line bundle* $\overline{N}$ *is viewed as an element of* $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ *by the natural pull-back map.*

(4) $\overline{M}$ *is* $\overline{L}$-bounded *if there is an* $\epsilon > 0$ *such that both* $\overline{L} + \epsilon\overline{M}$ *and* $\overline{L} - \epsilon\overline{M}$ *are nef.*

It is conventional that each of $\overline{H}_1 \leq \overline{H}_2$ and $\overline{H}_2 \geq \overline{H}_1$ means $\overline{H}_2 - \overline{H}_1 \geq 0$. Similar conventions apply to "$\equiv$" and "$\gg$." The main theorem of this paper is as follows.

Theorem 1.2. *Let* K *be a finitely generated field which is not a finite field, and* $\pi : X \to \mathrm{Spec}\, K$ *be a normal, geometrically connected, and projective variety of dimension* $n \geq 1$. *Let* $\overline{M}$ *be an integrable adelic line bundle on* X, *and* $\overline{L}_1, \cdots, \overline{L}_{n-1}$ *be* $n-1$ *nef line bundles on* X *where each* L_i *is big on* X. *Assume* $M \cdot L_1 \cdots L_{n-1} = 0$ *on* X. *Then*

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}) \leq 0.$$

Moreover, if $\overline{L}_i \gg 0$, *and* $\overline{M}$ *is* $\overline{L}_i$-bounded *for each* i, *then*

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}) \equiv 0$$

if and only if $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$.

1.2 Algebraic dynamics

Let X be a projective variety over a field K. A *polarizable algebraic dynamical system on X* is a morphism $f : X \to X$ such that there is an ample $\mathbb{Q}$-line bundle $L \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $f^*L = qL$ from some rational number $q > 1$. We call L a *polarization of f*, and call the triple (X, f, L) a *polarized algebraic dynamical system*. Let $\mathrm{Prep}(f)$ denote the set of *preperiodic points*, i.e.,

$$\mathrm{Prep}(f) := \{x \in X(\overline{K}) \mid f^m(x) = f^n(x) \text{ for some } m, n \in \mathbb{N}, \ m \neq n\}.$$

A well-known result of Fakhruddin [Fak] asserts that $\mathrm{Prep}(f)$ is Zariski dense in X.

Denote by $\mathcal{DS}(X)$ the set of all polarizable algebraic dynamical systems f on X. Note that we do *not* require elements of $\mathcal{DS}(X)$ to be polarizable by the same ample line bundle.

Theorem 1.3. *Let X be a projective variety over any field K. For any $f, g \in \mathcal{DS}(X)$, the following are equivalent:*

(1) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$;

(2) $g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)$;

(3) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense in X.

Remark 1.4. When $X = \mathbb{P}^1$, the theorem is independently proved by M. Baker and L. DeMarco [BD] during the preparation of this paper. They only wrote in characteristic zero, but their proof also applies to positive characteristics. Their treatment for the number field case is the same as our treatment in the earlier version, while the method for the general case is quite different.

Some consequences and questions related to the theorem can be found in [YZ, §3.3]. Here we only list the following one.

Theorem 1.5. *Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{C}_p$ for some prime number p. Let X be a projective variety over $\mathbb{K}$, and $f, g \in \mathcal{DS}(X)$ be two polarizable algebraic dynamical systems. If $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense in X, then $d\mu_f = d\mu_g$.*

Here $d\mu_f$ denotes the equilibrium measure of (X, f) on the Berkovich space X^{Ber} associated to X. It can be obtained from any initial "smooth" measure on X^{Ber} by Tate's limiting argument. By a proper interpretation, it satisfies $f^* d\mu_f = q^{\dim X} d\mu_f$ and $f_* d\mu_f = d\mu_f$.

The proof of Theorem 1.3 follows the idea in [YZ]. The Lefschetz principle allows us to assume that the base field K is a finitely generated field. The major input in the current case is the use of the notions of adelic line bundles over finitely generated fields, as a generalization of the theory in [Zh2]. For a polarized dynamical system (X, f, L) over a finitely generated field K, Tate's limiting argument gives an f-invariant adelic line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ extending L. Then we have a canonical height function

$$\mathfrak{h}_f : X(\overline{K}) \longrightarrow \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$$

defined by

$$\mathfrak{h}_f(x) = \frac{1}{\deg(x)} \pi_*(\overline{L}_f|_{\widetilde{x}}).$$

Here $\widetilde{x}$ denotes the closed point of X associated to x. This vector-valued height refines the canonical height of Moriwaki [Mo2, Mo3]. We further extend the notion of f-invariant adelic line bundles to that of f-admissible adelic line bundles as in [YZ]. With these preparations, we can apply the arithmetic Hodge index theorem to finish the proof.

As in an earlier draft of our paper, one may prove Theorem 1.3 without using our new theory of adelic line bundles over finitely generated fields. The proof combines the height theory of [Mo2, Mo3], the equidistribution idea of [SUZ] and [Yu], and the results of [YZ]. But the proof is very tricky and very technical due to many convergence problems of integrations and intersection numbers.

Acknowlegement

2 Intersection theory of adelic line bundles

In this section, we develop a theory of adelic line bundles over a finitely generated field K. We write all the details in the arithmetic case, and explain the geometric case briefly in the end.

2.1 Berkovich spaces and metrics

We extend the definition of Berkovich analytification of [Be] to flat schemes over $\mathbb{Z}$. It is the union of the original Berkovich spaces over all places of $\mathbb{Q}$.

We first set

$$(\operatorname{Spec} \mathbb{Q})^{\mathrm{an}} := \{\infty, 2, 3, 5, 7, \cdots\}.$$

It is the set of places of $\mathbb{Q}$, endowed with the discrete topology. Each $v \in (\operatorname{Spec} \mathbb{Q})^{\mathrm{an}}$ is identified with the normalized absolute value $|\cdot|_v$.

Let X be a scheme over $\mathbb{Q}$. We do not assume X to be of finite type. We define the *Berkovich space associated to X* to be

$$X^{\mathrm{an}} = \coprod_{v \in (\operatorname{Spec} \mathbb{Q})^{\mathrm{an}}} X_v^{\mathrm{an}}.$$

Here the definition and basic properties of of X_v^{an} and X^{an} are described as follows.

1. **Affine analytic space.** If X is covered by affine schemes $\operatorname{Spec} A$, then X_v^{an} as a set is covered by the affinoid $(\operatorname{Spec} A)_v^{\mathrm{an}}$ of multiplicative semi-norms on A which extends the absolute value $|\cdot|_v$ on $\mathbb{Q}$. For each $x \in (\operatorname{Spec} A)_v^{\mathrm{an}}$, the corresponding norm is a composition

$$A \longrightarrow \kappa_x \xrightarrow{|\cdot|} \mathbb{R}$$

where κ_x is the residue field and $|\cdot|$ is a valuation (multiplicative norm) on κ_x. We write the first map as $f \mapsto f(x)$. Then the image of f

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under the composition is written as $|f(x)|$, and we also write it as $|f|_x$ for convenience. Then the multiplicative semi-norm (on an affinoid) corresponding to $x \in X_v^{\mathrm{an}}$ is just $|\cdot|_x$.

2. **Topology.** The topology on X_v^{an} is the weakest one such that all $(\operatorname{Spec} A)_v^{\mathrm{an}}$ are open, and such that the function $|f(x)|$ is continuous for all f. The topology on X^{an} is induced by that on X_v^{an} via the disjoint union. If X is separated then X_v^{an} and X^{an} are Hausdorff.

3. **Canonical contraction and embeddings.** We have a continuous map $X^{\mathrm{an}} \to X$ by sending x to the kernel of the corresponding semi-norm on A. The fiber of a point is simply the the set of norms on the residue field of this point extending some normalized absolute value on $\mathbb{Q}$.

4. **Functoriality.** If $f : X \to Y$ is a morphism of flat $\mathbb{Q}$-schemes, then we get a morphism $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. If f is proper, so is f^{an}; if f has connected fibers, then f^{an} has arcwise connected fibers. If T is a subset of Y^{an} then we can define *the T-adic analytic space* by base change

$$f_T^{\mathrm{an}} : X_T^{\mathrm{an}} \longrightarrow T.$$

5. **Extension to $\mathbb{Z}$-schemes.** For any scheme $\mathcal{X}$ which is flat over $\mathbb{Z}$, for convenience, we set

$$\mathcal{X}^{\mathrm{an}} = (\mathcal{X}_{\mathbb{Q}})^{\mathrm{an}}, \quad \mathcal{X}_v^{\mathrm{an}} = (\mathcal{X}_{\mathbb{Q}})_v^{\mathrm{an}}.$$

Example 2.1. If X is of finite type over $\mathbb{Q}$ and v is non-archimedean, the space X_v^{an} is exactly the analytification $X_{\mathbb{Q}_v}^{\mathrm{Ber}}$ of the $\mathbb{Q}_v$-variety $X_{\mathbb{Q}_v}$ introduced by Berkovich [Be].

Example 2.2. For any scheme X over $\mathbb{Q}$,

$$X_\infty^{\mathrm{an}} = X(\mathbb{C})/\text{complex conjugation}$$

can be identified with the set of closed points of $X_{\mathbb{R}}$.

If X is integral with function field F, then $(\operatorname{Spec} F)^{\mathrm{an}}$ is identified with the subset of *generic points* on X^{an}, i.e., the fiber of the generic point of X under the map $X^{\mathrm{an}} \to X$. We have the following simple density result.

Lemma 2.3. *Let K be a finitely generated field over $\mathbb{Q}$, and let X be a variety over K with function field F. Then $(\operatorname{Spec} F)^{\mathrm{an}}$ is dense in X^{an}.*

Proof. We can extend the morphism $X \to \operatorname{Spec} K$ to a morphism $\mathcal{U} \to \mathcal{V}$ of finite type, where both $\mathcal{U}$ and $\mathcal{V}$ are varieties over $\mathbb{Q}$, and the function field of $\mathcal{V}$ is K. We have two injections $(\operatorname{Spec} F)^{\mathrm{an}} \to X^{\mathrm{an}}$ and $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$. It suffices to prove that $(\operatorname{Spec} F)^{\mathrm{an}}$ is dense in $\mathcal{U}^{\mathrm{an}}$. Equivalently, $(\operatorname{Spec} F)^{\mathrm{an}}_v$ is dense in $\mathcal{U}^{\mathrm{an}}_v$ for any place v of $\mathbb{Q}$. It is well-known since $\mathcal{U}^{\mathrm{an}}_v$ is the usual Berkovich space $\mathcal{U}^{\mathrm{Ber}}_{\mathbb{Q}_v}$ associated to $\mathcal{U}_{\mathbb{Q}_v}$. $\square$

Metrized line bundles

Let X be a $\mathbb{Q}$-scheme and L be a line bundle on X. It induces a line bundle L^{an} on X^{an}. At each point $x \in X^{\mathrm{an}}$, the fiber $L^{\mathrm{an}}(x)$ is the same as the fiber of L on the image of x in X. Then L^{an} is defined to be the union of all these fibers. By a *metric* $\|\cdot\|$ on L we mean a metric on L^{an} compatible with the semi-norms on $\mathcal{O}_X$. More precisely, to each point x in X^{an} with residue field κ we assign a norm $\|\cdot\|_x$ on the κ-line $L^{\mathrm{an}}(x)$ which is compatible with the norm $|\cdot|_x$ of κ in the sense that

$$\|f\ell\|_x = |f|_x \cdot \|\ell\|_x, \qquad f \in \kappa, \quad \ell \in L^{\mathrm{an}}(x).$$

We always assume that the metric $\|\cdot\|$ on L is *continuous* in that, for any section ℓ of L on a Zariski open subset U of X, the function $\|\ell(x)\| = \|\ell(x)\|_x$ is continuous on $x \in U^{\mathrm{an}}$.

We denote by $\operatorname{Pic}(X)$ (resp. $\mathcal{P}\mathrm{ic}(X)$) the Picard group of isomorphism classes (resp. the category) of line bundles on X. Let $\widehat{\operatorname{Pic}}(X^{\mathrm{an}})_{\mathrm{cont}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{cont}}$) denote the group of isometry classes (resp. the category) of line bundles on X endowed with continuous metrics. Thus we have natural morphisms

$$\widehat{\operatorname{Pic}}(X^{\mathrm{an}})_{\mathrm{cont}} \longrightarrow \operatorname{Pic}(X), \qquad \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{cont}} \longrightarrow \mathcal{P}\mathrm{ic}(X).$$

The fibers are homogeneous spaces of the group of metrics on $\mathcal{O}_X$.

If s is a rational section of a metrized line bundle with divisor D, then $-\log\|s\|$ defines a *Green's function* for the divisor D^{an} in the sense that this function is continuous outside the support of D^{an} and has logarithmic singularity along D^{an}. Conversely, given any divisor D of X and any Green's function g on X^{an} with logarithmic singularity along D^{an} in X^{an}, then e^{-g}

defines a metric on $\mathcal{O}(D)$. The pair (D, g) is called *an arithmetic divisor*. Let $\widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{cont}}$ denote the group of arithmetic divisors. Then we have just described a homomorphism

$$\widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{cont}} \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{cont}}.$$

Let $\phi : X \to Y$ be a morphism of $\mathbb{Q}$-schemes and M be a line bundle on Y with a metric on Y^{an}. Then one has a metric on $\phi^* M$ defined by the pull-back map.

For convenience, we also extend the definitions to flat $\mathbb{Z}$-schemes. If $\mathcal{X}$ flat $\mathbb{Z}$-scheme and $\mathcal{L}$ is a line bundle on $\mathcal{X}$, we have already set $\mathcal{X}^{\mathrm{an}} = (\mathcal{X}_{\mathbb{Q}})^{\mathrm{an}}$. Now set $\mathcal{L}^{\mathrm{an}} = (\mathcal{L}_{\mathbb{Q}})^{\mathrm{an}}$. Other definitions are extended similarly.

2.2 Integrable adelic line bundles

We need the following definitions:

1. **Arithmetic varieties.** By a *projective arithmetic variety (resp. open arithmetic variety)* we mean an integral scheme, projective (resp. quasi-projective) and flat over $\mathbb{Z}$. Resume all the positivity notions on projective arithmetic varieties in [YZ, §2.1].

2. **Projective model.** By a *projective model* of an open arithmetic variety $\mathcal{U}$ we mean an open embedding $\mathcal{U} \hookrightarrow \mathcal{X}$ into a projective arithmetic variety $\mathcal{X}$ such that the complement $\mathcal{X} \setminus \mathcal{U}$ is the support of an effective Cartier divisor.

3. **Arithmetic model.** Let K be a finitely generated field over $\mathbb{Q}$. By a *projective arithmetic model (resp. open arithmetic model)* of K we mean a projective arithmetic variety (resp. open arithmetic variety) with function field K. Let K be a finitely generated field over $\mathbb{Q}$ and X be a projective variety over K. By a *projective arithmetic model (resp. open arithmetic model)* of X/K we mean a projective and flat morphism $\mathcal{U} \to \mathcal{V}$ where:

   - $\mathcal{V}$ is a projective arithmetic model (resp. open arithmetic model) of K;
   - The generic fiber of $\mathcal{U} \to \mathcal{V}$ is $X \to \mathrm{Spec}\, K$.

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Let K be a finitely generated field over $\mathbb{Q}$, X a projective variety over K, and L a line bundle on X.

Let $\mathcal{X} \to \mathcal{B}$ be a projective model of X/K, and $\overline{\mathcal{L}}$ be a Hermitian line bundle on $\mathcal{X}$ with generic fiber $\overline{\mathcal{L}}_K$ isomorphic to L. This model induces a metric of $\mathcal{L}^{\mathrm{an}}$ on $\mathcal{X}^{\mathrm{an}}$, and thus a metric of L^{an} on X^{an}. We call the metric of L^{an} *the model metric induced by* $\overline{\mathcal{L}}$ and denote it by $\|\cdot\|_{\overline{\mathcal{L}}}$. Thus we have defined morphisms

$$\widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{cont}},$$

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{cont}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{cont}}.$$

These maps are *injective* since X^{an} is dense in $\mathcal{X}^{\mathrm{an}}$. The density is a consequence of Lemma 2.3.

$\mathcal{D}$-topology

Let $\mathcal{U}$ be an open arithmetic variety. Projective models $\mathcal{X}$ of $\mathcal{U}$ form a projective system. Using pull-back morphisms, we can form the direct limits:

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}},$$

$$\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}.$$

As usual, for each divisor $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$ we can construct an arithmetic line bundle $\mathcal{O}(\overline{\mathcal{D}})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$.

On a projective model $\mathcal{X}$, an arithmetic divisor $\overline{\mathcal{D}} = (\mathcal{D}, g) \in \widehat{\mathrm{Div}}(\mathcal{X})$ is *effective* (resp. *strictly effective*) if $\mathcal{D}$ is an effective (resp. strictly effective) divisor on $\mathcal{X}$ and the Green's function $g \geq 0$ (resp. $g > 0$) on $\mathcal{X}(\mathbb{C}) - |\mathcal{D}(\mathbb{C})|$. Consequently, a $\mathbb{Q}$-divisor $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ is called *effective* (resp. *strictly effective*) if for some positive integer n, the multiple $n\overline{\mathcal{D}}$ is an effective (resp. strictly effective) divisor in $\widehat{\mathrm{Div}}(\mathcal{X})$. For any $\overline{\mathcal{D}}, \overline{\mathcal{E}} \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$, we write $\overline{\mathcal{D}} > \overline{\mathcal{E}}$ or $\overline{\mathcal{E}} < \overline{\mathcal{D}}$ if $\overline{\mathcal{D}} - \overline{\mathcal{E}}$ is effective. It is a partial order on $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$, and compatible with pull-back morphisms. Thus we can talk about effectivity and the induced partial ordering on $\varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$.

Fix a projective model $\mathcal{X}$ and an effective Cartier divisor $\mathcal{D}$ with support $\mathcal{X} \setminus \mathcal{U}$. It induces a topology on $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$ as follows. Let g be any Green's function of $\mathcal{D}$ such that the arithmetic divisor $\overline{\mathcal{D}} = (\mathcal{D}, g) \in \widehat{\mathrm{Div}}(\mathcal{X})$ is strictly effective. Then a neighborhood basis at 0 is given by

$$B(\epsilon, \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}) := \{\overline{\mathcal{E}} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}} : \ -\epsilon\overline{\mathcal{D}} < \overline{\mathcal{E}} < \epsilon\overline{\mathcal{D}}\}, \quad \epsilon \in \mathbb{Q}_{>0}.$$

By translation, it gives a neighborhood basis at any point. The topology does not depend on the choice of $\mathcal{X}$ and $\mathcal{D}$. In fact, if $\overline{\mathcal{D}}' \in \widehat{\mathrm{Div}}(\mathcal{X}')_{\mathbb{Q}}$ is another effective divisor with support $\mathcal{X}' \setminus \mathcal{U}$, then we can find a third model dominating both $\mathcal{X}$ and $\mathcal{X}'$. Then we can find $r > 1$ such that $r^{-1}\overline{\mathcal{D}} < \overline{\mathcal{D}}' < r\overline{\mathcal{D}}$.

Adelic metrized line bundles

Let $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{cont}}$ be the completion of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$ and $\widehat{\mathrm{Pr}}(\mathcal{U})_{\mathrm{cont}}$ be the completion of the principal divisors. The group of *adelic line bundles on $\mathcal{U}$* is defined to be

$$\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}} = \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{cont}}/\widehat{\mathrm{Pr}}(\mathcal{U})_{\mathrm{cont}}.$$

The functor $\widehat{\mathrm{Pic}}(\cdot)_{\mathrm{cont}}$ is contravariant for projective morphisms.

Alternatively, we can define $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}}$ by the data $(\overline{\mathcal{L}}_i, \ell_{i,j})$ $(i \geq j \geq 0)$ with a convergence condition and an equivalent condition, where:

- $\overline{\mathcal{L}}_i$ is a sequence of line bundles on models $\mathcal{X}_i$ with compatible morphisms $\pi_{i,j} : \mathcal{X}_i \to \mathcal{X}_j$ $(i \geq j \geq 0)$ of $\mathcal{U}$-models, and

- $\{\ell_{i,j}\}_{i,j}$ is a compatible system of rational sections of $\mathcal{L}_i \otimes \pi_{i,j}^* \mathcal{L}_j^{-1}$ whose divisor supports on $\mathcal{X}_i \setminus \mathcal{U}$.

The convergence condition is that, for any $\epsilon > 0$, there is an i_0 such that for any $i \geq j \geq i_0$, the divisors

$$\epsilon\pi_{i,0}^* \widehat{D} \pm \widehat{\mathrm{div}}(\ell_{i,j})$$

are both strictly effective. A datum is equivalent to 0 if there are rational sections ℓ_i of $\overline{\mathcal{L}}_i$ such that:

- $\ell_{i,j} = \ell_i \otimes \pi_{i,j}^* \ell_j^{-1}$;

- for any $\epsilon > 0$ there is a i_0 such that for any $i \geq i_0$, the divisors

$$\epsilon \pi_{i,0}^* \widehat{D} \pm \widehat{\mathrm{div}}(\ell_i)$$

  are both strictly effective.

Definition 2.4. (1) We say that an adelic line bundle $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}}$ is *nef* (or equivalently *semipositive*) if it can be given by a Cauchy sequence $\{(\mathcal{X}_m, \overline{\mathcal{L}}_m)\}_m$ where each $\overline{\mathcal{L}}_m$ is nef.

(2) We say that a line bundle in $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}}$ is *integrable* if it is equal to the difference of two nef ones.

Denote by $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{nef}}$ and $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}}$ the subsets of nef elements and integrable elements respectively. Analogously, we introduce $\widehat{\mathcal{Pic}}(\mathcal{U})_{\mathrm{nef}}$ and $\widehat{\mathcal{Pic}}(\mathcal{U})_{\mathrm{int}}$ as subcategories of $\widehat{\mathcal{Pic}}(\mathcal{U})$.

Proposition 2.5. *The natural morphisms*

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}},$$

$$\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}},$$

$$\widehat{\mathcal{Pic}}(\mathcal{U})_{\mathrm{mod}} \longrightarrow \widehat{\mathcal{Pic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}}$$

can be extended continuously into morphisms

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{int}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{cont}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}},$$

$$\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}},$$

$$\widehat{\mathcal{Pic}}(\mathcal{U})_{\mathrm{mod}} \longrightarrow \widehat{\mathcal{Pic}}(\mathcal{U})_{\mathrm{int}} \longrightarrow \widehat{\mathcal{Pic}}(\mathcal{U})_{\mathrm{cont}} \longrightarrow \widehat{\mathcal{Pic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}}.$$

Proof. The image of $B(\epsilon, \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}})$ in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}}$ is the space of real-valune continuous functions on $\mathcal{U}^{\mathrm{an}}$ bounded by $\epsilon g_{\overline{D}^{\mathrm{an}}}$. Thus $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{cont}}$ is certainly complete with respect to this topology. This gives the extension of the first map. The other two follow from this one. $\square$

The map $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}} \to \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{cont}}$ is injective. Equivalently, the quotient $\overline{\mathcal{D}}$-topology on $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}}$ is separable. Namely, there is no nonzero element $\overline{\mathcal{E}} \in \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}}$ such that $-\epsilon \overline{\mathcal{D}} < \overline{\mathcal{E}} < \epsilon \overline{\mathcal{D}}$ for any $\epsilon > 0$. In fact, assume that some $\overline{\mathcal{E}} \in \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{mod}}$ satisfies the inequality. We can assume that $\overline{\mathcal{D}}$ and $\overline{\mathcal{E}}$

are realized on the same projective model $\mathcal{X}$. Denote $n = \dim \mathcal{U} - 1$. Then for any ample line bundles $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ in $\widehat{\mathrm{Pic}}(\mathcal{X})$, we have

$$(\epsilon \overline{\mathcal{D}} \pm \overline{\mathcal{E}}) \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n > 0, \quad \forall \epsilon > 0.$$

It follows that

$$\overline{\mathcal{E}} \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n = 0.$$

Then $\overline{\mathcal{E}} = 0$ by the Hodge index theorem for arithmetical divisors of Moriwaki [Mo1].

Remark 2.6. In [Fal], the definition of the Faltings height of an abelian variety uses the Petersson metric on the Hodge bundle of the (open) Siegel modular variety $\mathcal{A}_g$ over $\mathrm{Spec}\,\mathbb{Z}$. The metric has logarithmic singularity along the boundary $\mathcal{D}$. One can check that the corresponding metrized line bundle actually lies in $\widehat{\mathrm{Pic}}(\mathcal{A}_g)_{\mathrm{int}}$. Since $\mathcal{A}_g$ is only a stack, to make the statement rigorous one needs to put a level structure on it.

Relative case

Let K be a finitely generated field over $\mathbb{Q}$, and X be a projective variety over K. The set of open arithmetic models $\mathcal{U} \to \mathcal{V}$ of $X \to \mathrm{Spec}\,K$ form a projective system. Define

$$\begin{aligned}
\widehat{\mathrm{Div}}(X)_{\mathrm{int}} : &= \varinjlim_{\mathcal{U} \to \mathcal{V}} \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{int}}, \\
\widehat{\mathrm{Pic}}(X)_{\mathrm{int}} : &= \varinjlim_{\mathcal{U} \to \mathcal{V}} \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}}, \\
\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}} : &= \varinjlim_{\mathcal{U} \to \mathcal{V}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{int}}.
\end{aligned}$$

Then all these groups can be embedded into the corresponding objects on X^{an}. Notice that for any element $\overline{L}$ in $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}$, there are two models $(\mathcal{X}_1/\mathcal{B}_1, \overline{\mathcal{L}}_1)$ and $(\mathcal{X}_2/\mathcal{B}_2, \overline{\mathcal{L}}_2)$ with the same underlying bundle L such that

$$\| \cdot \|_{\overline{\mathcal{L}}_1} \leq \| \cdot \|_{\overline{L}} \leq \| \cdot \|_{\overline{\mathcal{L}}_2}.$$

If $X = \mathrm{Spec}(K)$, we also denote them by

$$\widehat{\mathrm{Div}}(K)_{\mathrm{int}}, \qquad \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}, \qquad \widehat{\mathcal{P}\mathrm{ic}}(K)_{\mathrm{int}}.$$

2.3 Arithmetic intersections

Let K be a finitely generated field over $\mathbb{Q}$ of transcendental degree d, and X be a projective variety over K of dimension n. In this section, we will introduce two intersection maps

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(K)_{\mathrm{int}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(K)_{\mathrm{int}}^{d+1} \longrightarrow \mathbb{R}.$$

They give definition of heights of subvarieties of X.

Deligne Pairing

Let $\pi : \mathcal{U} \to \mathcal{V}$ be a flat and projective morphism of open arithmetic varieties over $\mathrm{Spec}\,\mathbb{Z}$ of relative dimension n whose generic fiber is $X \to \mathrm{Spec}\,K$. Then we have an inductive family given by the Deligne pairing

$$\mathcal{P}\mathrm{ic}(\mathcal{U})^{n+1} \longrightarrow \mathcal{P}\mathrm{ic}(\mathcal{V}), \quad (\mathcal{L}_1, \cdots, \mathcal{L}_{n+1}) \longmapsto \langle \mathcal{L}_1, \mathcal{L}_2, \cdots \mathcal{L}_{n+1} \rangle.$$

We refer to [De] for the notion of the Deligne pairing.

Our goal is to extend this intersection to line bundles in $\widehat{\mathcal{P}\mathrm{ic}}(X)$. As in the case of [Zh2] and [Mo3], we can not expect the intersection to be defined for all adelic line bundles, but only for the integrable ones.

Proposition 2.7. *The above multilinear map extends to multilinear maps*

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{V})_{\mathrm{int}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(K)_{\mathrm{int}}.$$

Proof. By linearity, we only need to extend the image for semipositive line bundles $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{int}}$ for any open model $\pi : \mathcal{U} \to \mathcal{V}$ of $X \to \mathrm{Spec}\,K$.

For each $i = 1, \cdots, n+1$, suppose that $\overline{\mathcal{L}}_i$ is given by the Cauchy sequence $\{(\mathcal{X}_m, \overline{\mathcal{L}}_{i,m})\}_m$ with each $\overline{\mathcal{L}}_{i,m}$ ample on the model $\mathcal{X}_m$ over a projective model $\mathcal{B}_m$ of $\mathcal{V}$. Here we assume that the integral model $\mathcal{X}_m$ is independent of i, which is always possible. Apply Raynaud's flattening theorem in [Ra, Theorem 1, Chapter 4]. After blowing up $\mathcal{B}_m$ and replacing $\mathcal{X}_m$ by its pure transform, we can assume that $\pi_m : \mathcal{X}_m \to \mathcal{B}_m$ is flat. The goal is to show the convergence of

$$\langle \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \cdots, \overline{\mathcal{L}}_{n+1} \rangle := \lim_{m \to \infty} \langle \overline{\mathcal{L}}_{1,m}, \overline{\mathcal{L}}_{2,m}, \cdots, \overline{\mathcal{L}}_{n,m} \rangle.$$

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We can further assume that for each pair $m < m'$, the map $\pi_{m'}$ dominates π_m and

$$\overline{\mathcal{L}}_{i,m'} - \overline{\mathcal{L}}_{i,m} \simeq \mathcal{O}(Z_{i,m,m'}), \qquad \overline{Z}_{i,m,m'} \in B(\epsilon_m, \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}).$$

Here $\{\epsilon_m\}_{m \geq 1}$ is a sequence decreasing to zero.

We claim that for any $m < m'$,

$$\langle \overline{\mathcal{L}}_{1,m}, \overline{\mathcal{L}}_{2,m}, \cdots, \overline{\mathcal{L}}_{n+1,m} \rangle - \langle \overline{\mathcal{L}}_{1,m'}, \overline{\mathcal{L}}_{2,m'}, \cdots, \overline{\mathcal{L}}_{n+1,m'} \rangle \in B(\epsilon_m \deg X, \widehat{\mathcal{P}ic}(\mathcal{U})_{\mathrm{mod}}).$$

where

$$\deg(X) = \sum_{i=1}^{n} \deg(\mathcal{L}_{1,K} \cdot \mathcal{L}_{2,K} \cdots \mathcal{L}_{i-1,K} \cdot \mathcal{L}_{i+1,K} \cdots \mathcal{L}_{n+1,K}).$$

Then the sequence $\langle \overline{\mathcal{L}}_{1,m}, \overline{\mathcal{L}}_{2,m}, \cdots, \overline{\mathcal{L}}_{n+1,m} \rangle$ is Cauchy in $\widehat{\mathcal{P}ic}(\mathcal{U})_{\mathrm{mod}}$ under the $\mathcal{D}$-topology, and thus the convergence follows.

For the claim, note that

$$\langle \overline{\mathcal{L}}_{1,m}, \cdots, \overline{\mathcal{L}}_{n+1,m} \rangle - \langle \overline{\mathcal{L}}_{1,m'}, \cdots, \overline{\mathcal{L}}_{n+1,m'} \rangle$$
$$= \sum_{i=1}^{n} \langle \overline{\mathcal{L}}_{1,m}, \cdots, \overline{\mathcal{L}}_{i-1,m}, (\overline{\mathcal{L}}_{i,m} - \overline{\mathcal{L}}_{i,m'}), \overline{\mathcal{L}}_{i+1,m'}, \cdots, \overline{\mathcal{L}}_{n+1,m'} \rangle.$$

Fix an i and let ℓ_i be a section of $\overline{\mathcal{L}}_{i,m} - \overline{\mathcal{L}}_{i,m'}$ with divisor $\overline{Z}_{i,m,m'}$. Since all $\overline{\mathcal{L}}_{j,m}$ ($j < i$) and $\overline{\mathcal{L}}_{j,m'}$ ($j > i$) are ample, they have sections ℓ_j ($j \neq i$) so that $\mathrm{div}(\ell_j)$ for $1 \leq j \leq n+1$ intersects properly on $\mathcal{X}_{m'}$. In this way, the bundle

$$\langle \overline{\mathcal{L}}_{1,m}, \cdots, \overline{\mathcal{L}}_{i-1,m}, (\overline{\mathcal{L}}_{i,m} - \overline{\mathcal{L}}_{i,m'}), \overline{\mathcal{L}}_{i+1,m'}, \cdots, \overline{\mathcal{L}}_{n+1,m'} \rangle$$

has a section $\langle \ell_1, \cdots, \ell_{n+1} \rangle$ with divisor $\pi_*(\widehat{\mathrm{div}}\ell_1 \cdot \widehat{\mathrm{div}}\ell_2 \cdots \widehat{\mathrm{div}}\ell_{n+1})$. Since $\widehat{\mathrm{div}}(\ell_i) = \overline{Z}_{i,m,m'}$ is bounded by $\epsilon_m \pi^* \mathcal{D}$, where $\mathcal{D}$ is a fixed boundary divisor of $\mathcal{V}$, and since all other $\widehat{\mathrm{div}}(\ell_j)$ ($j \neq i$) are ample, we see that $\pi_*(\widehat{\mathrm{div}}\ell_1 \cdot \widehat{\mathrm{div}}\ell_2 \cdots \widehat{\mathrm{div}}\ell_{n+1})$ is bounded by

$$\epsilon_m \pi_* \left(\overline{\mathcal{L}}_{1,m} \cdots \overline{\mathcal{L}}_{i-1,m} \cdot \pi^* \overline{\mathcal{D}} \cdot \overline{\mathcal{L}}_{i+1,m'} \cdots \overline{\mathcal{L}}_{n+1,m'} \right)$$
$$= \epsilon_m \deg(\mathcal{L}_{1,K} \cdots \mathcal{L}_{i-1,K} \cdot \mathcal{L}_{i+1,K} \cdots \mathcal{L}_{n+1,K}) \, \overline{\mathcal{D}}.$$

It finishes the proof.

$\square$

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Intersection numbers

Let K be a field finitely generated over $\mathbb{Q}$ with transcendental degree d. For each projective model $\mathcal{X}$ of K, we have an intersection pairing $\widehat{\mathrm{Pic}}(\mathcal{X})^{d+1} \to \mathbb{R}$. It easily extends to a pairing $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}}^{d+1} \to \mathbb{R}$ for any open arithmetic model $\mathcal{U}$ of K.

Proposition 2.8. *The intersection pairing above extends uniquely to a multi-linear and continuous homomorphism*

$$\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}^{d+1} \longrightarrow \mathbb{R}.$$

Proof. It suffices to prove the similar result for $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}}$ for any open arithmetic variety $\mathcal{U}$ with function field K. We need to define $\langle \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{d+1} \rangle$ for any $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{d+1} \in \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int}}$.

Let $\mathcal{X}$ be a projective model of $\mathcal{U}$. Replacing $\mathcal{U}$ by an open subset if necessary, we may assume that $\mathcal{U}$ is the complement of an ample divisor $\mathcal{D}$ in $\mathcal{X}$. Complete $\mathcal{D}$ to an ample divisor $\overline{\mathcal{D}}$, and use it to define the $\mathcal{D}$-topology.

As in the proof in the previous proposition, we may assume that $\overline{\mathcal{L}}_i$ is given by a Cauchy sequence $\{(\mathcal{X}_m, \overline{\mathcal{L}}_{i,m})\}_m$ with each $\overline{\mathcal{L}}_{i,m}$ ample on a projective model $\mathcal{X}_m$ dominating $\mathcal{X}$. Assume for any $m' > m$,

$$\overline{\mathcal{L}}_{i,m'} - \overline{\mathcal{L}}_{i,m} \in B(\epsilon_m, \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}))$$

with $\epsilon_m \to 0$. For any subset $I \subset \{1, \cdots, d+1\}$, consider the sequence

$$\alpha_{I,m} := \overline{\mathcal{D}}^{d+1-|I|} \prod_{i \in I} \overline{\mathcal{L}}_{i,m}.$$

We want to prove by induction that $\{\alpha_{I,m}\}_m$ is a Cauchy sequence. When I is the full set, we have the proposition.

There is nothing to prove if I is an empty set. Assume the claim is true for any $|I| < k$ for some $k > 0$. Then for any I with $|I| = k$, we have

$$\overline{\mathcal{D}}^{d+1-k} \prod_{i \in I} \overline{\mathcal{L}}_{i,m} - \overline{\mathcal{D}}^{d+1-k} \prod_{i \in I} \overline{\mathcal{L}}_{i,m'}$$

$$= \overline{\mathcal{D}}^{d+1-k} \sum_{i \in I} \prod_{j \in I, j < i} \overline{\mathcal{L}}_{j,m} (\overline{\mathcal{L}}_{i,m} - \overline{\mathcal{L}}_{i,m'}) \prod_{j \in I, j > i} \overline{\mathcal{L}}_{j,m'}.$$

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Its absolutely value is bounded by

$$\epsilon_m \overline{\mathcal{D}}^{d+1-k+1} \sum_{i\in I} \prod_{j\in I, j<i} \overline{\mathcal{L}}_{j,m} \prod_{j\in I, j>i} \overline{\mathcal{L}}_{j,m'}$$
$$\leq \epsilon_m \overline{\mathcal{D}}^{d+1-k+1} \sum_{i\in I} \prod_{j\in I, j<i} \overline{\mathcal{L}}_{j,m} \prod_{j\in I, j>i} (\overline{\mathcal{L}}_{j,m} + \epsilon_m \overline{D}).$$

The last term is a linear combination of $\alpha_{I',m}$ with $|I'| < k$. The coefficients of the linear combination grow as $o(\epsilon_m)$. It follows that $\alpha_{I,m}$ is a Cauchy sequence.

$\square$

Recall that an element $\overline{H}$ of $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ is said to be *numerically trivial* if

$$\overline{H} \cdot \overline{N}_1 \cdots \overline{N}_d = 0, \quad \forall \, \overline{N}_1, \cdots, \overline{N}_d \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}.$$

One can prove that the subgroup of numerically trivial elements of $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ is exactly $\widehat{\mathrm{Pic}}^0(F)$, viewed as a natural subgroup of $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ via pull-back. Here F denotes the algebraic closure of $\mathbb{Q}$ in K, which is a number field, and $\widehat{\mathrm{Pic}}^0(F)$ denotes the kernel of the degree map $\widehat{\deg} : \widehat{\mathrm{Pic}}(F) \to \mathbb{R}$. But we do not need this fact in the paper.

2.4 Arithmetic heights

In this section, we introduce a vector-valued height function, which refines the Moriwaki height in [Mo2, Mo3]. The Northcott property and the theorem of successive minima are deduced from those of Moriwaki.

Let K be a field finitely generated over $\mathbb{Q}$ of transcendental degree d and X be a projective variety over K of dimension n. Let $\overline{L}$ be a nef element in $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$ with ample generic fiber L. For any closed $\overline{K}$-subvariety Z of X, define *the arithmetic height of Z with respect to $\overline{L}$* as

$$\mathfrak{h}_{\overline{\mathcal{L}}}(Z) := \frac{\langle \overline{L}|_{\widetilde{Z}} \rangle^{\dim Z + 1}}{(\dim Z + 1) \deg_L(\widetilde{Z})} \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}.$$

Here $\widetilde{Z}$ denotes the minimal K-subvariety of X containing Z, $\overline{L}|_{\widetilde{Z}}$ denotes the *pull-back* in $\widehat{\mathrm{Pic}}(\widetilde{Z})_{\mathrm{int}}$, and the self-intersections are taken as the Deligne pairing in the sense of Proposition 2.7. It gives a map

$$\mathfrak{h}_{\overline{L}} : |X_{\overline{K}}| \longrightarrow \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}.$$

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Here $|X_{\overline{K}}|$ denotes the set of closed $\overline{K}$-subvarieties of X. In particular, we have a height function of algebraic points:

$$\mathfrak{h}_{\overline{L}} : X(\overline{K}) \longrightarrow \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}.$$

The class of $\mathfrak{h}_{\overline{L}}$ modulo bounded functions depends only on the class of L in $\mathrm{Pic}(X)$.

If K is a number field, we have the degree map $\deg : \widehat{\mathrm{Pic}}(K) \to \mathbb{R}$. In that case, $\deg \mathfrak{h}_{\overline{L}}$ is the same as the usual height.

Moriwaki heights

Let $K, X, \overline{L}$ be as above. Let $\overline{H}_1, \cdots, \overline{H}_d$ be any d elements in $\widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$. For any closed $\overline{K}$-subvariety Z of X, *the Moriwaki height of Z with respect to $\overline{L}$ and $(\overline{H}_1, \cdots, \overline{H}_d)$* is

$$h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_d}(Z) := \mathfrak{h}_{\overline{L}}(Z) \cdot \overline{H}_1 \cdots \overline{H}_d = \frac{\langle \overline{L}|_{\widetilde{Z}} \rangle^{\dim Z + 1} \cdot \overline{H}_1 \cdots \overline{H}_d}{(\dim Z + 1) \deg_L(\widetilde{Z})}.$$

It gives a real-valued function

$$h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_d} : |X_{\overline{K}}| \longrightarrow \mathbb{R}.$$

In the case $\overline{H}_1 = \cdots = \overline{H}_d = \overline{H}$, the height is written as $h_{\overline{L}}^{\overline{H}}$.

If both $\overline{L}$ and $(\overline{H}_1, \cdots, \overline{H}_d)$ are realized on some projective model $\mathcal{X} \to \mathcal{B}$ of $X \to \mathrm{Spec}\, K$, then $(\mathcal{B}, \overline{H}_1, \cdots, \overline{H}_d)$ is called a polarization of K, and $h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_d}$ is exactly the height function introduced in [Mo2]. In [Mo3], Moriwaki generalizes the definition to the case that $\overline{L}$ is given by an adelic sequences. Our notion of adelic line bundle includes Moriwaki's adelic sequences.

If $\overline{L}, \overline{H}_1, \cdots, \overline{H}_d$ are induced by integral models, and $\overline{H}_1, \cdots, \overline{H}_d$ are nef and big, Moriwaki [Mo2] proves that the height satisfies the Northcott property. Namely, for any $D \in \mathbb{R}$ and $A \in \mathbb{R}$, the set

$$\{x \in X(\overline{K}) : \deg(x) < D, \; h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_d}(x) < A\}$$

is finite.

The theorem of successive minima of Zhang [Zh1, Zh2] is generalized by Moriwaki [Mo2]. We say an adelic line bundle $\overline{H} \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ satisfies *the*

Moriwaki condition, if it is induced by a nef Hermitian line bundle on a projective model of K, and the top self-intersection number $\overline{H}^{d+1} = 0$. If furthermore $\overline{L}$ is induced by a nef Hermitian line bundles, then Moriwaki's result asserts that

$$\lambda_1^{\overline{H}}(X, \overline{L}) \geq h_{\overline{L}}^{\overline{H}}(X).$$

Here the essential minimum

$$\lambda_1^{\overline{H}}(X, \overline{L}) = \sup_{U \subset X} \inf_{x \in U(\overline{K})} h_{\overline{L}}^{\overline{H}}(x),$$

where the supremum is taken over all Zariski open subsets U of X.

Consequences

As consequences, we have the following Northcott property and the theorem of successive minima.

Theorem 2.9 (Northcott property). *Let $\overline{L}$ be an element in $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ with ample generic fiber L. For any $D \in \mathbb{R}$ and $\alpha \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$, the set*

$$\{x \in X(\overline{K}) : \ \deg(x) < D, \ \mathfrak{h}_{\overline{L}}(x) \leq \alpha\}$$

is finite.

Note that $\mathfrak{h}_{\overline{L}}(x) \leq \alpha$ is the partial order defined by pseudo-effectivity. To convert the result to the current case, take any big and nef $\overline{H}_1, \cdots, \overline{H}_d$ induced by integral models. The key is that $h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_d} - h_{\overline{L}^0}^{\overline{H}_1, \cdots, \overline{H}_d}$ is a bounded function on $X(\overline{K})$ as long as the underlying line bundle of $\overline{L}^0$ is also L. Then we can assume that $\overline{L}$ is also induced by an integral model.

Theorem 2.10 (successive minima). *Let $\overline{L}$ be a nef element in $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$ with ample generic fiber L. Let $\overline{H}$ be an element of $\widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$ satisfying the Moriwaki condition. Then*

$$\lambda_1^{\overline{H}}(X, \overline{L}) \geq h_{\overline{L}}^{\overline{H}}(X).$$

The new part of the theorem is that $\overline{L}$ is not necessarily induced by an integral model. We can approximate $\overline{L}$ by a sequence of nef Hermitian line bundles on integral models. In the process, $h_{\overline{L}}^{\overline{H}}(x)$ is approximated uniformly in x. The result follows.

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2.5 Geometric case

Let k be any field, K be a finitely generated field over k, and X be a projective variety over K. We may extend all the definitions and results to this setting, depending on the choice of k.

The definition of the adelic line bundles is already given in §1. More precisely, we define

$$\widehat{\mathrm{Div}}(X/k)_{\mathrm{int}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}}, \quad \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int}}$$

to be the injective limits of

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{int}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{int}}, \quad \widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{int}}$$

over all open models $\mathcal{U} \to \mathcal{V}$ of $X \to \mathrm{Spec}\, K$, where $\mathcal{U}$ and $\mathcal{V}$ quasi-projective varieties over k, and $\mathcal{U} \to \mathcal{V}$ is required to be projective and flat.

We do not assume that k is a finite field here, though it is necessary in Theorem 1.2 and for the Northcott property. If k is a finite field, we abbreviate the dependence on k, and write the objects as

$$\widehat{\mathrm{Div}}(X)_{\mathrm{int}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}, \quad \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}.$$

The positivity notions and intersection theory are defined similarly in this setting. To define the Berkovich space interpreting the adelic line bundles, we need to choose an intermediate field k_1 of K/k which is separable and finitely generated over k of transcendental degree 1. It exists if the transcendental degree of K over k is at least 1.

Then we define $(\mathrm{Spec}\, k_1)^{\mathrm{an}}$ to be the set of places of k_1 over k endowed with the discrete topology. Each place v is endowed with a normalized absolute $|\cdot|_v$. Note that k_1 is the function field of a projective and smooth curve C over k. Then $(\mathrm{Spec}\, k_1)^{\mathrm{an}}$ is just the set of closed points of C.

Let X be a variety over K as above. We define the *Berkovich space associated to X/k_1* to be

$$X^{\mathrm{an}} = (X/k_1)^{\mathrm{an}} = \coprod_{v \in (\mathrm{Spec}\, k_1)^{\mathrm{an}}} X_v^{\mathrm{an}}.$$

Here X_v^{an} is described as follows. If X is covered by affine schemes $\mathrm{Spec}\, A$, then $X_v^{\mathrm{an}} = (X/k_1)_v^{\mathrm{an}}$ as a set is covered by the affinoid $(\mathrm{Spec}\, A)_v^{\mathrm{an}}$ of multiplicative semi-norms on A which extends the absolute value $|\cdot|_v$ on k_1. The

topology on X_v^{an} is the weakest one such that all $(\mathrm{Spec}\, A)_v^{\mathrm{an}}$ are open, and such that the function $|f(x)|$ is continuous on X_v^{an} for all f. The topology on X^{an} is induced by that on X_v^{an} via the disjoint union.

Then we can define line bundles on X with continuous metrics over the analytic space X^{an} similarly to the arithmetic case. It follows that we have a lot of injective homomorphisms like

$$\widehat{\mathrm{Pic}}(X)_{\mathrm{int}} \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{cont}}.$$

3   Arithmetic Hodge index theorem

In this section, we are going to prove Theorem 1.2. We will give a detailed proof for the arithmetic case, and sketch a proof for the geometric case in the end.

3.1   The inequality

We first deduce the inequality of Theorem 1.2 from the main theorem of [YZ]. By approximation, it suffices to prove the following assertion.

Let $\pi : \mathcal{X} \to \mathcal{B}$ be a projective and flat morphism of projective arithmetic varieties. Write $\dim \mathcal{B} = d+1$ and $\dim \mathcal{X} = n+d+1$. Let $\overline{\mathcal{M}}$ be a Hermitian line bundle on $\mathcal{X}$, $(\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n-1})$ be nef Hermitian line bundles on $\mathcal{X}$ with big generic fibers on X, and $(\overline{\mathcal{H}}_1, \cdots, \overline{\mathcal{H}}_d)$ be nef Hermitian line bundle on $\mathcal{B}$. Assume that the generic fiber $\mathcal{L}_{i,\eta}$ is big on the generic fiber $\mathcal{X}_\eta$ of $\mathcal{X}$ above the generic point η of $\mathcal{B}$ for every $i = 1, \cdots, n-1$. If

$$\mathcal{M}_\eta \cdot \mathcal{L}_{1,\eta} \cdots \mathcal{L}_{n-1,\eta} = 0,$$

then

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \cdot \pi^*\overline{\mathcal{H}}_1 \cdots \pi^*\overline{\mathcal{H}}_d \leq 0.$$

We follow the idea at the beginning of [YZ, §2.4]. We can assume that each $\overline{\mathcal{H}}_j$ is ample on $\mathcal{B}$ since nef bundles are limits of ample line bundles. For simplicity, denote $\overline{\mathcal{L}}_{n-1+j} = \pi^*\overline{\mathcal{H}}_j$ for $j = 1, \cdots, d$. Fix an ample Hermitian line bundle $\overline{\mathcal{A}}$ on $\mathcal{X}$. Take a small rational number $\epsilon > 0$. Set $\overline{\mathcal{M}}' = \overline{\mathcal{M}} + \delta\overline{\mathcal{A}}$ and $\overline{\mathcal{L}}_i' = \overline{\mathcal{L}}_i + \epsilon\overline{\mathcal{A}}$ for $i = 1, \cdots d+n-1$. Here δ is a number such that

$$\mathcal{M}'_{\mathbb{Q}} \cdot \mathcal{L}'_{1,\mathbb{Q}} \cdots \mathcal{L}'_{d+n-1,\mathbb{Q}} = (\mathcal{M}_{\mathbb{Q}} + \delta\mathcal{A}_{\mathbb{Q}}) \cdot \mathcal{L}'_{1,\mathbb{Q}} \cdots \mathcal{L}'_{d+n-1,\mathbb{Q}} = 0.$$

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It determines
$$\delta = -\frac{\mathcal{M}_{\mathbb{Q}} \cdot \mathcal{L}'_{1,\mathbb{Q}} \cdots \mathcal{L}'_{d+n-1,\mathbb{Q}}}{\mathcal{A}_{\mathbb{Q}} \cdot \mathcal{L}'_{1,\mathbb{Q}} \cdots \mathcal{L}'_{d+n-1,\mathbb{Q}}}.$$

As $\epsilon \to 0$, we have $\delta \to 0$ since

$$\mathcal{M}_{\mathbb{Q}} \cdot \mathcal{L}'_{1,\mathbb{Q}} \cdots \mathcal{L}'_{d+n-1,\mathbb{Q}} \longrightarrow \mathcal{M}_{\mathbb{Q}} \cdot \mathcal{L}_{1,\mathbb{Q}} \cdots \mathcal{L}_{d+n-1,\mathbb{Q}}$$
$$= (\mathcal{M}_\eta \cdot \mathcal{L}_{1,\eta} \cdots \mathcal{L}_{n-1,\eta})(\mathcal{H}_{1,\mathbb{Q}} \cdots \mathcal{H}_{d,\mathbb{Q}}) = 0$$

and

$$\mathcal{A}_{\mathbb{Q}} \cdot \mathcal{L}'_{1,\mathbb{Q}} \cdots \mathcal{L}'_{d+n-1,\mathbb{Q}} \longrightarrow \mathcal{A}_{\mathbb{Q}} \cdot \mathcal{L}_{1,\mathbb{Q}} \cdots \mathcal{L}_{d+n-1,\mathbb{Q}}$$
$$= (\mathcal{A}_\eta \cdot \mathcal{L}_{1,\eta} \cdots \mathcal{L}_{n-1,\eta})(\mathcal{H}_{1,\mathbb{Q}} \cdots \mathcal{H}_{d,\mathbb{Q}}) > 0.$$

The last inequality uses the assumption that $\mathcal{L}_{i,\eta}$ is big and nef for each i.

Applying the main theorem of [YZ] to the arithmetic variety $\mathcal{X}$ over $\mathbb{Z}$, we have
$$\overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}'_1 \cdots \overline{\mathcal{L}}'_{d+n-1} \leq 0.$$

Set $\epsilon \to 0$. We have
$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d+n-1} \leq 0.$$

It proves the result.

3.2 Equality: vertical case

An adelic line bundle $\overline{L} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ is called *vertical* if the underlying line bundle L is trivial on X. Denote by $\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}}$ the group of vertical adelic line bundles on X.

Now we prove the condition of the equality of the theorem in the vertical case. Recall that:

- K is a finitely generated field over $\mathbb{Q}$ of transcendental degree $d \geq 0$;

- X is a normal projective variety of dimension $n \geq 1$ over K;

- $\overline{M} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}}$ is vertical;

- $\overline{L}_1, \cdots, \overline{L}_{n-1} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ with $\overline{L}_i \gg 0$;

- $\overline{M}$ is $\overline{L}_i$-bounded for each i;

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- The equality
$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}) \equiv 0$$
holds on $\operatorname{Spec} K$.

We need to prove $\overline{M} \in \pi^* \widehat{\operatorname{Pic}}(K)_{\mathrm{int}}$.

By definition, there is an open model $\mathcal{U} \to \mathcal{V}$ of $X \to \operatorname{Spec} K$ such that

$$\overline{M}, \overline{L}_1, \cdots, \overline{L}_{n-1} \in \widehat{\operatorname{Pic}}(\mathcal{U})_{\mathrm{int}}.$$

For any horizontal closed integral subscheme $\mathcal{W}$ of $\mathcal{V}$ of dimension $e + 1$, we get a projective and flat morphism $\mathcal{U}_{\mathcal{W}} \to \mathcal{W}$, and it defines the groups $\widehat{\operatorname{Pic}}(\mathcal{U}_{\mathcal{W}})_{\mathrm{int}}$ and $\widehat{\operatorname{Pic}}(\mathcal{W})_{\mathrm{int}}$. There are natural pull-back maps

$$\widehat{\operatorname{Pic}}(\mathcal{U})_{\mathrm{int}} \to \widehat{\operatorname{Pic}}(\mathcal{U}_{\mathcal{W}})_{\mathrm{int}}, \quad \widehat{\operatorname{Pic}}(\mathcal{V})_{\mathrm{int}} \to \widehat{\operatorname{Pic}}(\mathcal{W})_{\mathrm{int}}.$$

We first prove the following result.

Lemma 3.1. *For any $\overline{H}_1, \cdots, \overline{H}_e \in \widehat{\operatorname{Pic}}(\mathcal{V})_{\mathrm{int}}$, one has*

$$(\overline{M}|_{\mathcal{U}_{\mathcal{W}}})^2 \cdot (\overline{L}_1|_{\mathcal{U}_{\mathcal{W}}}) \cdots (\overline{L}_{n-1}|_{\mathcal{U}_{\mathcal{W}}}) \cdot (\overline{H}_1|_{\mathcal{W}}) \cdots (\overline{H}_e|_{\mathcal{W}}) = 0.$$

Proof. By induction, we can assume that $\mathcal{W}$ has codimension one in $\mathcal{V}$. We need to prove

$$(\overline{M}|_{\mathcal{U}_{\mathcal{W}}})^2 \cdot (\overline{L}_1|_{\mathcal{U}_{\mathcal{W}}}) \cdots (\overline{L}_{n-1}|_{\mathcal{U}_{\mathcal{W}}}) \cdot (\overline{H}_1|_{\mathcal{W}}) \cdots (\overline{H}_{d-1}|_{\mathcal{W}}) = 0.$$

By approximation, we can assume that there is a projective model $\mathcal{X} \to \mathcal{B}$ of $\mathcal{U} \to \mathcal{V}$ such that $\overline{H}_i \in \widehat{\operatorname{Pic}}(\mathcal{B})$ for every $i = 1, \cdots, d-1$. Denote by $\mathcal{C}$ the Zariski closure of $\mathcal{W}$ in $\mathcal{B}$. Then $\mathcal{X}_{\mathcal{C}} \to \mathcal{C}$ is a projective model of $\mathcal{U}_{\mathcal{W}} \to \mathcal{W}$.

By assumption, for any $\overline{H}_d \in \widehat{\operatorname{Pic}}(\mathcal{B})$, we have $\mathcal{I} \cdot \overline{H}_d = 0$. Here we note

$$\mathcal{I} = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \cdot \overline{H}_1 \cdots \overline{H}_{d-1}.$$

Then the intersection of $\mathcal{I}$ with any vertical class of $\mathcal{B}$ is zero. Now assume that the finite part $\mathcal{H}_d$ of $\overline{H}_d$ is ample on $\mathcal{B}_{\mathbb{Q}}$. After replacing $\overline{H}_d$ by a multiple if necessary, we can assume that there is a section s of $\mathcal{H}_d$ vanishing on $\mathcal{C}$. It follows that we can write

$$\operatorname{div}(s) = \sum_{i=0}^{r} a_i \mathcal{C}_i, \quad a_i \geq 0.$$

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Here $\mathcal{C}_0 = \mathcal{C}$ and $a_0 > 0$. By definition of intersetion numbers,

$$\mathcal{I} \cdot \overline{H}_d = \sum_{i=0}^{r} a_i \mathcal{I} \cdot \mathcal{C}_i - \int_{\mathcal{B}(\mathbb{C})} \log \|s\| \omega_{\mathcal{I}}.$$

Here the integral is a formal intersection of $\log \|s\|$ with $\mathcal{I}$, which is zero since $\mathcal{I}$ has zero intersection with any vertical class. Furthermore, $\mathcal{I} \cdot \mathcal{C}_i = 0$ if $\mathcal{C}_i$ is vertical, and $\mathcal{I} \cdot \mathcal{C}_i \le 0$ by the inequality part of Theorem 1.2. Hence, $\mathcal{I} \cdot \overline{H}_d = 0$ forces $\mathcal{I} \cdot \mathcal{C} = 0$. It is exactly the equality that we need to prove. $\quad\square$

Set $\dim \mathcal{W} = 1$ in the lemma. Then the function field of $\mathcal{W}$ is a number field. Apply the main theorem of [YZ], we conclude that

$$\overline{M}|_{\mathcal{U}_{\mathcal{W}}} \in \pi^* \widehat{\mathrm{Pic}}(\mathcal{W})_{\mathrm{int}}.$$

To imply $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$, we first re-interpret it in terms of Berkovich spaces.

Recall that we have injections

$$\widehat{\mathrm{Pic}}(X)_{\mathrm{cont}} \hookrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{cont}}, \quad \widehat{\mathrm{Pic}}(K)_{\mathrm{cont}} \hookrightarrow \widehat{\mathrm{Pic}}((\mathrm{Spec}\, K)^{\mathrm{an}})_{\mathrm{cont}}.$$

We claim that $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ is equivalent to $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}((\mathrm{Spec}\, K)^{\mathrm{an}})_{\mathrm{cont}}$. In fact, assume the later. If there is a rational point $s \in X(K)$, then we would have $\overline{M} = \pi^* \overline{M}_0$ where $\overline{M}_0 = s^* \overline{M}$ lies in $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$. The identity can be checked in $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{cont}}$. In general, taking any $x \in X(\overline{K})$, we have $\overline{M} = \pi^* \mathfrak{h}_{\overline{M}}(x)$ with $\mathfrak{h}_{\overline{M}}(x) \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$.

Hence, it suffices to prove $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}((\mathrm{Spec}\, K)^{\mathrm{an}})_{\mathrm{cont}}$. Since M is trivial, the metric of $\overline{M}$ corresponds to a continuous function

$$-\log \|1\|_{\overline{M}} : X^{\mathrm{an}} \longrightarrow \mathbb{R}.$$

It suffices to prove that $\log \|1\|_{\overline{M}}$ is constant on the fiber of any point of $(\mathrm{Spec}\, K)^{\mathrm{an}}$.

Let $\mathcal{U} \to \mathcal{V}$ and $\mathcal{W}$ be as above. Then $\log \|1\|_{\overline{M}}$ extends to $\mathcal{U}^{\mathrm{an}}$. By $\overline{M}|_{\mathcal{U}_{\mathcal{W}}} \in \pi^* \widehat{\mathrm{Pic}}(\mathcal{W})_{\mathrm{int}}$, we see that $\log \|1\|_{\overline{M}}$ is constant on the fibers of $\mathcal{U}^{\mathrm{an}} \to \mathcal{V}^{\mathrm{an}}$ above w_v for any closed point w of $\mathcal{V}_{\mathbb{Q}}$ and any place v of $\mathbb{Q}$. Here w_v denotes the finite subset of classical points of $\mathcal{V}_v^{\mathrm{an}}$ corresponding to the finite subset of closed points $\mathcal{V}_{\mathbb{Q}_v}$ determined by w. By the density of $\{w_v\}_w$ in $\mathcal{V}_{\mathbb{Q}_v}^{\mathrm{an}}$, we conclude that $\log \|1\|_{\overline{M}}$ is constant on any fiber of $\mathcal{U}^{\mathrm{an}} \to \mathcal{V}^{\mathrm{an}}$. Then it is constant on any fiber of $X^{\mathrm{an}} \to (\mathrm{Spec}\, K)^{\mathrm{an}}$. It finishes the proof.

3.3 Equality: case of curves

When X is a curve, we present a theorem which interprets the intersection numbers in terms of the Neron–Tate height. It generalizes the result of Faltings [Fal] and Hriljac [Hr] to finitely generated fields. It implies Theorem 1.2 for curves easily.

The height identity

Let K be a finitely generated field over $\mathbb{Q}$ of transcendental degree d, and let $\pi : X \to \operatorname{Spec} K$ be a smooth, projective, and geometrically connected curve of genus $g > 0$. We first introduce the canonical height function

$$\hat{\mathfrak{h}} : \operatorname{Pic}^0(X_{\overline{K}}) \longrightarrow \widehat{\operatorname{Pic}}(K)_{\mathrm{int}}.$$

Denote by $J = \underline{\operatorname{Pic}}^0(X)$ the Jacobian variety of X. Denote by Θ the symmetric line bundle on J associated to the theta divisor. Namely, choose a point $x_0 \in X(\overline{K})$ and denote by $j : X_{\overline{K}} \hookrightarrow J_{\overline{K}}$ the embedding $x \mapsto [x - x_0]$. Denote by θ the image of the composition $X_{\overline{K}}^{g-1} \hookrightarrow J_{\overline{K}}^{g-1} \to J_{\overline{K}}$. The second map is the sum under the group law. Then θ is a divisor of $J_{\overline{K}}$. Denote by Θ the line bundle on $J_{\overline{K}}$ associated to $\theta + [-1]^*\theta$. One checks that the isomorphism class of Θ does not depend on the choice of x_0, and Θ descend to a line bundle on J.

By the symmetric and ample line bundle Θ on J, we have the canonical height

$$\hat{\mathfrak{h}}_\Theta : J(\overline{K}) \longrightarrow \widehat{\operatorname{Pic}}(K)_{\mathrm{int}}.$$

By convention, we set $\hat{\mathfrak{h}} = \dfrac{1}{2}\hat{\mathfrak{h}}_\Theta$.

Theorem 3.2. *Let K be a finitely generated field over $\mathbb{Q}$ of transcendental degree d, and let $\pi : X \to \operatorname{Spec} K$ be a smooth, projective, and geometrically connected curve. Let M be a line bundle on X with $\deg M = 0$. Then the following are true:*

(1) There is an adelic line bundle $\overline{M}_0 \in \widehat{\operatorname{Pic}}(X)_{\mathrm{int}}$ with underlying line bundle M such that $\pi_*(\overline{M}_0 \cdot \overline{V}) \equiv 0$ for any $\overline{V} \in \widehat{\operatorname{Pic}}(X)_{\mathrm{vert}}$;

(2) $\overline{M}_0$ is unique up to translation by $\pi^*\widehat{\operatorname{Pic}}(K)_{\mathrm{int}}$;

(3) $\pi_*(\overline{M}_0 \cdot \overline{M}_0) \equiv -2\,\widehat{\mathfrak{h}}(M)$.

The theorem implies Theorem 1.2 for curves easily. In fact, define $\overline{N} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}}$ by
$$\overline{M} = \overline{M}_0 + \overline{N}.$$
Note that $\pi_*(\overline{M}_0 \cdot \overline{N}) \equiv 0$. We have
$$\pi_*(\overline{M} \cdot \overline{M}) \equiv \pi_*(\overline{M}_0 \cdot \overline{M}_0) + \pi_*(\overline{N} \cdot \overline{N}) \leq 0.$$
Here $\pi_*(\overline{M}_0 \cdot \overline{M}_0) \equiv -2\,\widehat{\mathfrak{h}}(M) \leq 0$ by Theorem 3.2 and $\pi_*(\overline{N} \cdot \overline{N}) \leq 0$ by the vertical case of Theorem 1.2.

If the equality holds, then $\widehat{\mathfrak{h}}(M) = 0$. It implies that M is torsion. Replacing M by a multiple if necessary, we can assume that M is trivial. By the vertical case of Theorem 1.2, we conclude that $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$.

The proof of Theorem 3.2(2) is also immediate. In fact, if $\overline{M}_0$ and $\overline{M}_0'$ are two different extensions satisfying the property. Then the difference $\overline{V} = \overline{M}_0 - \overline{M}_0'$ is vertical and thus perpendicular to both $\overline{M}_0$ and $\overline{M}_0'$. It follows that $\overline{V}^2 = 0$. Then $\overline{V} \in \pi^* \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ by the vertical case of Theorem 1.2.

The universal bundle

Here we prove Theorem 3.2. The proof is written almost the same as the number field case. We include it here briefly. For basic geometric results on abelian varieties and Jacobian varieties, we refer to [Mu] and [Se].

Let P be a universal bundle on $X \times J$. Namely, for any $\alpha \in J(\overline{K})$, the restriction $P|_{X \times \alpha}$ is isomorphic to the line bundle on X represented by α. Then P is determined by this property up to translation by $p_2^* \mathrm{Pic}(J)$. Here $p_1 : X \times J \to X$ and $p_2 : X \times J \to J$ denotes the projections. Via p_1, we view $X \times J$ as an abelian scheme over X. Denote by $[m]_X : X \times J \to X \times J$ the multiplication by an integer m as abelian schemes over X, i.e., the product of the identity map on the first component and the multiplication by m on the second component.

If there is a rational point $x_0 \in X(\overline{K})$, we rigidify P by setting $P|_{x_0 \times J}$ to be trivial. Then P is anti-symmetric. In general, we have $[-1]_X^* P = -P + C$ for some $C \in p_2^* \mathrm{Pic}(J)$. Then we can achieve $[-1]_X^* P = -P$ by replacing P by $P - \frac{1}{2}C$.

In any case, we have $[2]_X^* P = 2P$ on $X \times J$. By Tate's limiting argument, we have an extension $\overline{P} \in \widehat{\mathrm{Pic}}(X \times J)_{\mathrm{int}}$ of P such that $[2]_X^* \overline{P} = 2\overline{P}$.

Now we can prove part (1) of the theorem. Let α be the point of J representing the line bundle $M \in \mathrm{Pic}^0(X)$. Set

$$\overline{M}_0 = \overline{P}|_{X \times \alpha} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}.$$

We claim that $\overline{M}_0$ satisfies the requirement of (1).

In fact, we can prove the universal property that, for any $\overline{V} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}}$, the line bundle

$$\overline{R} = p_{2,*}(\overline{P} \cdot p_1^* \overline{V})$$

is 0 in $\widehat{\mathrm{Pic}}(J)_{\mathrm{int}}$. In fact, the underlying line bundle $R = 0$ since $V = 0$. We further have $[2]_X^* \overline{R} = 2\overline{R}$. It must be 0. It proves Theorem 3.2 (1).

To prove part (3), we first express the intersection as a height function. Replacing the field K by a finite extension if necessary, we can assume that $j : X \to J$ is defined by a base point $x_0 \in X(K)$. Rigidify P by $P_{x_0 \times J} = 0$ as above. The universal line bundle P on $X \times J$ can be extended to the Poincare line bundle on $J \times J$, which we still denote by P. It is symmetric on $J \times J$. Thus $[2]^* P = 4P$ and we can extend it to $\overline{P} \in \widehat{\mathrm{Pic}}(J \times J)_{\mathrm{int}}$. It is compatible with the adelic line bundle $\overline{P}$ originally defined on $X \times J$.

We remark that $[2]^* P = 4P$ does not contradict to the original $[2]_X^* P = 2P$. In fact, $[2] : J \times J \to J \times J$ is multiplication by 2 on both components, while $[2]_X : X \times J \to X \times J$ is only multiplication by 2 on the second component.

Lemma 3.3. *For any $\alpha, \beta \in J(K)$, we have*

$$\pi_*(\overline{P}_\alpha \cdot \overline{P}_\beta) = \mathfrak{h}_{\overline{P}}(\alpha, \beta).$$

Here $\overline{P}_\alpha = \overline{P}|_{X \times \alpha}$ and $\overline{P}_\beta = \overline{P}|_{X \times \beta}$ are viewed as adelic line bundles on X.

Proof. Note both sides are bilinear in (α, β). We can assume that α represents the divisor $x - x_0$ on X. Then $\alpha = j(x)$. Here we assume $x \in X(K)$ by replacing K by a finite extension if necessary. Then we have

$$\pi_*(\overline{P}_\alpha \cdot \overline{P}_\beta) = \pi_*((\hat{x} - \hat{x}_0) \cdot \overline{P}_\beta) = \pi_*(\hat{x} \cdot \overline{P}_\beta) - \pi_*(\hat{x}_0 \cdot \overline{P}_\beta).$$

Here $\hat{x}$ and $\hat{x}_0$ are any extensions of x and x_0 in $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$. Note that $\overline{P}_\beta$ has zero intersection with any vertical classes. The above becomes

$$\pi_*(\overline{P}|_{x \times \beta}) - \pi_*(\overline{P}|_{x_0 \times \beta}) = \pi_*(\overline{P}|_{x \times \beta}) = \mathfrak{h}_{\overline{P}}(\alpha, \beta).$$

$\square$

Now we are ready to prove part (3) of the theorem. By the lemma, it suffices to prove $\mathfrak{h}_{\overline{P}}(\alpha, \alpha) = -\mathfrak{h}_{\overline{\Theta}}(\alpha)$ for any $\alpha \in J(K)$. It is well known that the Poincare bundle on $J \times J$ has the expression

$$P = p_1^*\theta + p_2^*\theta - m^*\theta.$$

Here $m, p_1, p_2 : J \times J \to J$ denotes the addition law and the projections. It induces

$$2P = p_1^*\Theta + p_2^*\Theta - m^*\Theta.$$

We use Θ because it is also symmetric. It follows that

$$2\overline{P} = p_1^*\overline{\Theta} + p_2^*\overline{\Theta} - m^*\overline{\Theta}.$$

Computing heights using the identity, we have

$$2\mathfrak{h}_{\overline{P}}(\alpha, \alpha) = \mathfrak{h}_{\overline{\Theta}}(\alpha) + \mathfrak{h}_{\overline{\Theta}}(\alpha) - \mathfrak{h}_{\overline{\Theta}}(2\alpha) = -2\mathfrak{h}_{\overline{\Theta}}(\alpha).$$

It proves part (3).

3.4 Equality: general case

The proof of the equality part of Theorem 1.2 is almost identical to that in [YZ]. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following.

Argument on the generic fiber

Assume the conditions in the equality part of the Theorem 1.2, which particularly includes

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}) \equiv 0.$$

We first show that M is numerically trivial on X by the condition $\overline{L}_{n-1} \gg 0$.

The condition asserts that $\overline{L}'_{n-1} = \overline{L}_{n-1} - \overline{N}$ is nef for some $\overline{N} \in \widehat{\mathrm{Pic}}(\mathbb{Q})$ with $\widehat{\deg}(\overline{N}) > 0$. Then

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}_{n-1}) = \pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1}) + (M^2 \cdot L_1 \cdots L_{n-2})\overline{N}.$$

Applying the inequality of the theorem to $(\overline{M}, \overline{L}_1, \cdots, \overline{L}_{n-2}, \overline{L}'_{n-1})$, we have

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1}) \leq 0.$$

By the Hodge index theorem on X in the geometric case, we have

$$M^2 \cdot L_1 \cdots L_{n-2} \leq 0.$$

Hence,

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1}) \equiv 0, \quad M^2 \cdot L_1 \cdots L_{n-2} = 0.$$

On the variety X, we have

$$M \cdot L_1 \cdots L_{n-2} \cdot L_{n-1} = 0, \quad M^2 \cdot L_1 \cdots L_{n-2} = 0.$$

By the Hodge index theorem on normal algebraic varieties, we conclude that M is numerically trivial. See the appendix of [YZ].

Numerically trivial case

We have proved that M is numerically trivial on X, and now we continue to prove that M is a torsion line bundle. Then a multiple of $\overline{M}$ is vertical and has already been treated. As in [YZ], the key is still the variational method.

Lemma 3.4. *Let $\overline{M}, \overline{L}_1, \cdots, \overline{L}_{n-1}$ be integrable adelic line bundles on X such that the following conditions hold:*

(1) M is numerically trivial on X;

(2) $\overline{M}$ is $\overline{L}_i$-bounded for every i;

(3) $\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}) \equiv 0$.

For any nef adelic line bundles $\overline{L}_i^0$ on X with underlying bundle L_i^0 numerically equivalent to L_i, and any integrable adelic line bundle $\overline{M}'$ with numerically trivial underlying line bundle M', the following are true:

$$\pi_*(\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0) \equiv 0,$$

$$\pi_*(\overline{M}^2 \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-2}^0) \equiv 0.$$

Proof. The proof is similar to its counterpart in [YZ]. For example for the first equality, it suffices to prove

$$\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 \cdot \pi^* \overline{H}_1 \cdots \pi^* \overline{H}_d = 0$$

for any nef $\overline{H}_1, \cdots, \overline{H}_d \in \widehat{\mathrm{Pic}}(\mathrm{Spec}\, K)_{\mathrm{int}}$. For fixed $\overline{H}_1, \cdots, \overline{H}_d$, the intersection numbers still satisfy the Cauchy–Schwartz inequality. The proof can be carried here. $\square$

Go back to the equality part of Theorem 1.2. Apply Bertini's theorem. Replacing $\overline{L}_{n-1}$ by a positive multiple if necessary, there is a section $s \in H^0(X, L_{n-1})$ such that $Y = \operatorname{div}(s)$ is an integral subvariety of X, regular in codimension one. Then we have

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}_{n-1}) \equiv \pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y).$$

In fact, the difference of two sides is the limit of the intersection of $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2}$ with vertical classes, so it vanishes by the second equality of the lemma. Hence,

$$\pi_*(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y) \equiv 0.$$

By the Lefschetz hyperplane theorem, we can assume that $\operatorname{Pic}^0(X)_{\mathbb{Q}} \to \operatorname{Pic}^0(Y)_{\mathbb{Q}}$ is injective. It reduces the problem to Y. The proof is complete since we have already treated the case of curves.

3.5 Geometric case

In the geometric case, Theorem 1.2 can be proved similarly. The only result we have used in characteristic 0 which is not known in positive characteristics is the existence of resolution of singularities, but the purpose to use it in characteristic 0 is to treat archimedean places. So it is not needed in the geometric case.

In the vertical case, we still use induction on the transcendental degree $d+1$ of K over the finite field k. The initial case is $d+1 = 1$, where K itself is the function field of a curve over k. The result in this case can be proved as in the arithmetic case in [YZ].

4 Algebraic dynamics

In this section, we first develop a theory of admissible adelic line bundles for polarizable algebraic dynamical systems over finitely generated fields, following the idea of [Zh2, YZ]. Then we prove Theorem 1.3.

4.1 Invariant adelic line bundles

Let (X, f, L) be a *polarized dynamical system* over a field K, i.e.,

- X is a projective variety over K;

- $f : X \to X$ is a morphism over K;

- $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ is an ample line bundle such that $f^*L = qL$ from some $q > 1$.

If K is a number field, by [Zh2], Tate's limiting argument gives an adelic $\mathbb{Q}$-line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ extending L and with $f^*\overline{L}_f = q\overline{L}_f$. In the following we generalize the definition to finitely generated fields.

In the following, assume that K is finitely generated field over $\mathbb{Q}$ with transcendental degree d. Fix an isomorphism $f^*L = qL$ where $q > 1$ by assumption.

Invariant adelic line bundle

Fix a projective modle $\mathcal{B}$ of K. Choose any arithmetic model $\pi : \mathcal{X} \to \mathcal{B}$ over $\mathcal{B}$ and any Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ over $\mathcal{X}$ such that $(\mathcal{X}_K, \mathcal{L}_K) = (X, L)$.

For each positive integer m, consider the composition $X \xrightarrow{f^m} X \hookrightarrow \mathcal{X}$. Denote its normalization by $f_m : \mathcal{X}_m \to \mathcal{X}$, and the induced map to $\mathcal{B}$ by $\pi_m : \mathcal{X}_m \to \mathcal{B}$. Denote $\overline{\mathcal{L}}_m = q^{-m} f_m^* \overline{\mathcal{L}}$, which lies in $\widehat{\mathrm{Pic}}(\mathcal{X}_m)_{\mathbb{Q}}$. The sequence $\{(\mathcal{X}_m, \overline{\mathcal{L}}_m)\}_{m \geq 1}$ is an adelic structure in the sense of [Mo3].

In the following, we will show that the sequence $\{(\mathcal{X}_m, \overline{\mathcal{L}}_m)\}_{m \geq 1}$ converges to a line bundle $\overline{L}_f$ in $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$. There is an open subscheme $\mathcal{V}$ of $\mathcal{B}$ such that $\mathcal{U} := \mathcal{X}_{\mathcal{V}}$ is flat over $\mathcal{V}$ and that $f : X \to X$ extends to a morphism $f_{\mathcal{V}} : \mathcal{U} \to \mathcal{U}$ with $f_{\mathcal{V}}^* \mathcal{L}_{\mathcal{V}} = q\mathcal{L}_{\mathcal{V}}$. By the construction, we have $\mathcal{X}_{m,\mathcal{V}} = \mathcal{X}_{\mathcal{V}}$ and $\mathcal{L}_{m,\mathcal{V}} = \mathcal{L}_{\mathcal{V}}$. We can assume that $\mathcal{D} := \mathcal{B} - \mathcal{V}$ is an effective Cartier divisor of $\mathcal{B}$ by enlarging $\mathcal{V}$ if necessary.

Theorem 4.1. *The sequence $\overline{L}_f = \{(\mathcal{X}_m, \overline{\mathcal{L}}_m)\}_{m \geq 1}$ is convergent in $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$. Furthermore, the limit is nef and depends only on the generic fiber (X, f, L, K).*

Proof. We only prove the existence of the limit, since the independence of the integral models can be proved similarly. Recall that $\mathcal{X}_{m,\mathcal{V}} = \mathcal{X}_{\mathcal{V}}$ and $\mathcal{L}_{m,\mathcal{V}} = \mathcal{L}_{\mathcal{V}}$. But these isomorphisms are not given by the morphism $f_m : \mathcal{X}_m \to \mathcal{X}$.

Let $\widetilde{\pi}_m : \widetilde{\mathcal{X}}_m \to \mathcal{B}$ be an arithmetic model of X which dominates both $\mathcal{X}_m$ and $\mathcal{X}_{m+1}$. More precisely, $\widetilde{\pi}_m$ factors through a birational morphism $\phi_m : \widetilde{\mathcal{X}}_m \to \mathcal{X}_m$ (resp. $\phi'_m : \widetilde{\mathcal{X}}_m \to \mathcal{X}_{m+1}$) such that $\phi_{m,\mathcal{V}}$ (resp. $\phi'_{m,\mathcal{V}}$) is an isomorphism to $\mathcal{X}_{m,\mathcal{V}}$ (resp. $\mathcal{X}_{m+1,\mathcal{V}}$). The construction works for $m = 0$ by the convention $\mathcal{X}_0 = \mathcal{X}$.

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We first consider the relation between $(\mathcal{X}, \overline{\mathcal{L}})$ and $(\overline{\mathcal{L}}_1, \mathcal{X}_1)$. The difference $\phi_0^* \overline{\mathcal{L}} - \phi_0'^* \overline{\mathcal{L}}_1$ is an arithmetic $\mathbb{Q}$-line bundle on $\widetilde{\mathcal{X}}_0$. It is trivial on $\widetilde{\mathcal{X}}_{0,\mathcal{V}}$, and thus represented by an arithmetic divisor supported on $\widetilde{\pi}_0^* |\mathcal{D}|$. Then there exists $r > 0$ such that

$$\phi_0^* \overline{\mathcal{L}} - \phi_0'^* \overline{\mathcal{L}}_1 \in B(r, \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}})$$

Now we consider $\overline{\mathcal{L}}_m - \overline{\mathcal{L}}_{m+1}$ for general m. Without loss of generality, we can assume that $\widetilde{\mathcal{X}}_m$ dominates $\widetilde{\mathcal{X}}_0$ via a birational morphism $\tau_m : \widetilde{\mathcal{X}}_m \to \widetilde{\mathcal{X}}_0$. Then it is easy to see that

$$\phi_m^* \overline{\mathcal{L}}_m - \phi_m'^* \overline{\mathcal{L}}_{m+1} = \frac{1}{q^m} \tau_m^* (\phi_0^* \overline{\mathcal{L}} - \phi_0'^* \overline{\mathcal{L}}_1).$$

It follows that

$$\phi_m^* \overline{\mathcal{L}}_m - \phi_m'^* \overline{\mathcal{L}}_{m+1} \in B(\frac{r}{q^m}, \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}})$$

In terms of the partial order in $\widehat{\mathrm{Pic}}(X)_{\mathrm{mod}}$, it is just

$$\overline{\mathcal{L}}_m - \overline{\mathcal{L}}_{m+1} \in B(\frac{r}{q^m}, \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}).$$

It follows that $\{\overline{\mathcal{L}}_m\}_m$ is a Cauchy sequence. $\qquad\square$

By construction, the arithmetic line bundle $\overline{L}_f$ is invariant under the pullback $f^* : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(X)$ in the sense that $f^* \overline{L}_f = q \overline{L}_f$ as in the number field case.

Canonical height

Let K and (X, f, L) be as above. It gives an f-invariant line bundle $\overline{L}_f$ in $\widehat{\mathrm{Pic}}(X)$. For any closed $\overline{K}$-subvariety Z of X, define *the canonical height function of Z* as

$$\mathfrak{h}_f(Z) = \mathfrak{h}_{L,f}(Z) := \mathfrak{h}_{\overline{L}_f}(Z) \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}.$$

It gives a map $\mathfrak{h}_f : |X_{\overline{K}}| \to \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$.

We can also define the canonical height by Tate's limiting argument:

$$\mathfrak{h}_f(Z) = \lim_{m \to \infty} \frac{1}{q^m} \mathfrak{h}_{(\mathcal{X}, \overline{\mathcal{L}})}(f^m(Z)).$$

Here $(\mathcal{X}, \overline{\mathcal{L}})$ is any initial model of (X, L) as in the construction of $\overline{L}_f$ above. Then one can check that it is convergent in $\widehat{\mathrm{Pic}}(K)$ and compatible with the previous definition.

Proposition 4.2. *Let Z be a closed subvariety of X over $\overline{K}$. Then:*

(1) The height $\mathfrak{h}_f(Z)$ lies in $\widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$.

(2) The height is f-invariant in the sense that $\mathfrak{h}_f(f(Z)) = q\,\mathfrak{h}_f(Z)$.

(3) The height $\mathfrak{h}_f(Z) = 0$ in $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ if Z is preperiodic under f. The inverse is also true if Z is a point.

Proof. Since $\overline{L}_f$ is nef, the height $\mathfrak{h}_f(Z)$ is nef. The formula $\mathfrak{h}_f(f(Z)) = q\mathfrak{h}_f(Z)$ follows from the projection formula and the invariance of $\overline{L}_f$. Thus $\mathfrak{h}_f(Z) = 0$ if Z is preperiodic under f. The second statement of (3) follows from the Northcott property. $\square$

By choosing adelic line bundles $\overline{H}_1, \cdots, \overline{H}_d \in \widehat{\mathcal{P}\mathrm{ic}}(K)_{\mathrm{nef}}$, we can form *the Moriwaki canonical height*

$$h_f^{\overline{H}_1, \cdots, \overline{H}_d}(Z) := \mathfrak{h}_f(Z) \cdot \overline{H}_1 \cdots \overline{H}_d.$$

Neron–Tate height

Let X be an abelian variety over K, $f = [2]$ be the multiplication by 2, and L be any symmetric and ample line bundle. Then the canonical height

$$\widehat{\mathfrak{h}}_L = \mathfrak{h}_{L,[2]} : X(\overline{K}) \longrightarrow \widehat{\mathrm{Pic}}(K)_{\mathrm{nef}},$$

as a generalization of the Neron–Tate height, is quadratic in that

$$\langle x, y \rangle_L := \widehat{\mathfrak{h}}_L(x + y) - \widehat{\mathfrak{h}}_L(x) - \widehat{\mathfrak{h}}_L(y)$$

gives a bilinear map

$$X(\overline{K}) \times X(\overline{K}) \longrightarrow \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}.$$

It can be proved by the theorem of the cube as in the classical case over number fields. We refer to [Se] for the classical case.

4.2 Admissible adelic line bundles

Let (X, f, L) be a polarized dynamical system over a finitely generated field K over $\mathbb{Q}$. Assume that X is normal. We have already construted an adelic line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$ extending L and with $f^*\overline{L}_f = q\overline{L}_f$. Following the idea of [YZ], we can construct an admissible extension for any line bundle $M \in \mathrm{Pic}(X)$. Our exposition is sketchy, and we refer to [YZ] for more details.

Semisimplicity

The pull-back map f^* preserves the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

It is known that $\mathrm{NS}(X)$ is a finitely generated $\mathbb{Z}$-module. Then $\mathrm{Pic}^0(X)$ is also a finitely generated $\mathbb{Z}$-module, since it is the Mordell–Weil group of the Picard variety representing the functor $\underline{\mathrm{Pic}}^0(X)$ over the finitely generated field K. The counterpart of [YZ, Theorem 3.1] is as follows.

Theorem 4.3. *(1) The operator f^* is semisimple on $\mathrm{Pic}^0(X)_{\mathbb{C}}$ (resp. $\mathrm{NS}(X)_{\mathbb{C}}$) with eigenvalues of absolute values $q^{1/2}$ (resp. q).*

(2) The operator f^* is semisimple on $\mathrm{Pic}(X)_{\mathbb{C}}$ with eigenvalues of absolute values $q^{1/2}$ or q.

The proof is similar to its counterpart. The only difference is that, we need to use the Moriwaki height to define a negative definite pairing on $\mathrm{Pic}^0(X)$. Then the proof goes through by Theorem 3.2.

By the theorem above, the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X)_{\mathbb{C}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{C}} \longrightarrow \mathrm{NS}(X)_{\mathbb{C}} \longrightarrow 0.$$

has a splitting

$$\ell_f : \mathrm{NS}(X)_{\mathbb{C}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{C}}$$

by identifying $\mathrm{NS}(X)_{\mathbb{C}}$ with the subspace of $\mathrm{Pic}(X)_{\mathbb{C}}$ generated eigenvectors whose eigenvalues have absolute values q. It is easy to see that the splitting actually descends to

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}.$$

Definition 4.4. We say an element of $\mathrm{Pic}(X)_{\mathbb{C}}$ is *f-pure of weight* 1 *(resp. 2)* if it lies in $\mathrm{Pic}^0(X)_{\mathbb{C}}$ *(resp. $\ell_f(\mathrm{NS}(X)_{\mathbb{C}})$).*

Admissible extensions

Recall that $\widehat{\mathrm{Pic}}(X) = \widehat{\mathrm{Pic}}(X)_{\mathrm{cont}}$ is the group of adelic line bundles on X. It is already a $\mathbb{Q}$-vertor space. Assume that $\pi : X \to \mathrm{Spec}\,K$ is geometrically connected. Write $\mathbf{F}$ for $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. We introduce

$$\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]} := \frac{\widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Q}} \mathbf{F}}{(\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Q}} \mathbf{F})_0}.$$

Here we describe the subspace in the denominator. Denote by $C(X^{\mathrm{an}}, \mathbf{F})$ the space of continuous functions from X^{an} to $\mathbf{F}$. If $\mathbf{F} = \mathbb{Q}$, we endow it with the discrete topology. The map $\log\|1\| : \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \to C(X^{\mathrm{an}}, \mathbb{R})$ extends to an $\mathbf{F}$-linear map

$$\log\|1\| : \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Q}} \mathbf{F} \to C(X^{\mathrm{an}}, \mathbf{F}).$$

Define

$$(\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Q}} \mathbf{F})_0 := \ker(\log\|1\| : \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Q}} \mathbf{F} \to C(X^{\mathrm{an}}, \mathbf{F})).$$

By definition,

$$\widehat{\mathrm{Pic}}(X)_{[\mathbb{Q}]} = \widehat{\mathrm{Pic}}(X), \quad \widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]} = \widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]} \otimes_{\mathbb{R}} \mathbb{C}.$$

Define $\widehat{\mathrm{Pic}}(X)_{\mathrm{int},[\mathbf{F}]}$ to be the image of $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}} \otimes_{\mathbb{Q}} \mathbf{F}$ in $\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$. The intersection theory extends to $\widehat{\mathrm{Pic}}(X)_{\mathrm{int},[\mathbf{F}]}$ by linearity. The positivity notions are extended to $\widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]}$.

The action $f^* : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(X)$ extends to $\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$ naturally. The goal is to study the spectral theory of this action.

Definition 4.5. An element $\overline{M}$ of $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$ is called f-*admissible* if we can write $\overline{M} = \sum_{i=1}^{m} \overline{M}_i$ such that each $\overline{M}_i$ is an eigenvector of f^* in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$.

The first main result asserts the existence of an admissible section of the forgetful map $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]} \to \mathrm{Pic}(X)_{\mathbb{C}}$.

Theorem 4.6. *For any $M \in \mathrm{Pic}(X)_{\mathbb{C}}$, there exists a unique f-admissible lifting $\overline{M}_f$ of M in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$. Moreover, for $\mathbf{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, if $M \in \mathrm{Pic}(X)_{\mathbf{F}}$, then $\overline{M}_f \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int},[\mathbf{F}]}$.*

The second main result here is the following positivity result.

Theorem 4.7. *If $M \in \mathrm{Pic}(X)_{\mathbb{R}}$ is ample and f-pure of weight 2, then $\overline{M}_f$ is nef.*

Next we introduce the natural section of the projection

$$\widehat{\mathrm{Pic}}(X) \longrightarrow \mathrm{NS}(X)_{\mathbb{Q}}.$$

Definition 4.8. For $\mathbf{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, define

$$\widehat{\ell}_f : \mathrm{NS}(X)_{\mathbf{F}} \to \widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$$

to be the map which sends $\xi \in \mathrm{NS}(X)_{\mathbf{F}}$ to the unique f-admissible class in $\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$ extending $\ell_f(\xi)$.

4.3   Preperiodic points

The goal of this section is to prove Theorem 1.3. By Lefschetz principle, we can assume that K is finitely generated over $\mathbb{Q}$ or a finite field k. Note that the theorem is trivial if K is a finite field, so we assume that K is infinite. The following result refines the theorem. The condition of X being normal can be obtained by taking a normalization.

Theorem 4.9. *Let X be a normal projective variety over a finitely generated field K. For any $f, g \in \mathcal{DS}(X)$, the following are equivalent:*

(1) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$;

(2) $g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)$;

(3) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense in X;

(4) $\widehat{\ell}_f = \widehat{\ell}_g$ as maps from $\mathrm{NS}(X)_{\mathbb{Q}}$ to $\widehat{\mathrm{Pic}}(X)$.

We only prove the arithmetic case that K is finitely generated over $\mathbb{Q}$, leaving the readers to figure out the details in the case that K is finitely generated over a finite field.

As in the number field case, we prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$. The proofs of the easy directions are similar to the number field case. In the implication $(2) \Rightarrow (3)$, we need the finiteness of

$$\mathrm{Prep}(f, r) := \{x \in \mathrm{Prep}(f) \mid \deg(x) < r\}.$$

It is given by Northcott's property of the Moriwaki canonical height. In the following, we prove the hard implication $(3) \Rightarrow (4)$.

Applying the Hodge index theorem

Assume that $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense in X. As usual, write n for the dimension of X and d for the transcendental degree of K over $\mathbb{Q}$. We need to prove $\widehat{\ell}_f(\xi) = \widehat{\ell}_g(\xi)$ for any $\xi \in \mathrm{NS}(X)_{\mathbb{Q}}$. By linearity, it suffices to assume that ξ is ample.

Denote $L = \ell_f(\xi)$ and $M = \ell_g(\xi)$. They are ample $\mathbb{Q}$-line bundles on X. Then $\overline{L}_f = \widehat{\ell}_f(\xi)$ and $\overline{M}_g = \widehat{\ell}_g(\xi)$ are nef by Theorem 4.7.

Consider the sum $\overline{N} = \overline{L}_f + \overline{M}_g$, which is still nef. By Theorem 2.10,

$$\lambda_1^{\overline{H}}(X, \overline{N}) \geq h_{\overline{N}}^{\overline{H}}(X)$$

for any $\overline{H} \in \widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$ satisfying the Moriwaki condition. Note that the essential minimum $\lambda_1^{\overline{H}}(X, \overline{N}) = 0$ since $h_{\overline{N}}^{\overline{H}}$ is zero on $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$, which is assumed to be Zariski dense in X. It forces $h_{\overline{N}}^{\overline{H}}(X) = 0$. Write in terms of intersections, we have

$$(\overline{L}_f + \overline{M}_g)^{n+1} \cdot \overline{H}^d = 0.$$

Expand by the binomial formula. Note that every term is non-negative. It follows that

$$\overline{L}_f^i \cdot \overline{M}_g^{n+1-i} \cdot \overline{H}^d = 0, \quad \forall i = 0, 1, \cdots, n+1.$$

It is true for any $\overline{H}$ satisfying the Moriwaki condition. We can remove the dependence on $\overline{H}$ by the following result.

Lemma 4.10. *Let $\overline{Q} \in \widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$ be a nef adelic line bundle such that the intersection number $\overline{Q} \cdot \overline{H}^d = 0$ for any $\overline{H} \in \widehat{\mathrm{Pic}}(K)_{\mathrm{nef}}$ satisfying the Moriwaki condition. Then $\overline{Q}$ is numerically trivial.*

We will prove the lemma later. With the lemma, we have

$$\pi_*(\overline{L}_f^i \cdot \overline{M}_g^{n+1-i}) \equiv 0, \quad \forall i = 0, 1, \cdots, n+1.$$

Then the proof is similar to the number field case. In fact, we have

$$\pi_*((\overline{L}_f - \overline{M}_g)^2 \cdot (\overline{L}_f + \overline{M}_g)^{n-1}) \equiv 0.$$

We still have

$$(L - M) \cdot (L + M)^{n-1} = 0$$

since $L - M \in \text{Pic}^0(X)_{\mathbb{Q}}$ is numerically trivial. Apply Theorem 1.2 to

$$(\overline{L}_f - \overline{M}_g, \ \overline{L}_f + \overline{M}_g).$$

It is trivial that $(\overline{L}_f - \overline{M}_g)$ is $(\overline{L}_f + \overline{M}_g)$-bounded. To meet the condition $\overline{L}_f + \overline{M}_g \gg 0$, we can take any $\overline{C} \in \widehat{\text{Pic}}(\mathbb{Q})$ with $\deg(\overline{C}) > 0$, and replace

$$(\overline{L}_f - \overline{M}_g, \ \overline{L}_f + \overline{M}_g)$$

by

$$(\overline{L}_f - \overline{M}_g, \ \overline{L}_f + \overline{M}_g + \pi^*\overline{C}).$$

Then all the conditions are satisfied. The theorem implies that

$$\overline{L}_f - \overline{M}_g \in \pi^*\widehat{\text{Pic}}(K)_{\text{int}}.$$

By evaluating at any point x in $\text{Prep}(f) \cap \text{Prep}(g)$, we see that

$$\overline{L}_f - \overline{M}_g = 0$$

in $\widehat{\text{Pic}}(X)_{\mathbb{Q}}$. It proves the theorem.

Moriwaki condition

It remains to prove Lemma 4.10. Assume $\overline{Q} \in \widehat{\text{Pic}}(\mathcal{V})_{\text{int}}$ for some open model $\mathcal{V}$ of K. We prove the lemma by a few steps.

Step 1. Replacing $\mathcal{V}$ by an open subscheme if necessary, the height function on $\mathcal{V}(\overline{\mathbb{Q}})$ associated to $\overline{Q}$ is identically 0. Namely, for any horizontal closed integral subscheme $\mathcal{W}$ of dimension one in $\mathcal{V}$, the restriction $\overline{Q}|_{\mathcal{W}} \in \widehat{\text{Pic}}(\mathcal{W})_{\text{int}}$ has arithmetic degree 0.

By Noether's normalization lemma, we can assume that there is a finite morphism $\psi : \mathcal{V} \to \mathcal{V}_0$ for some open subscheme $\mathcal{V}_0$ of $\mathbb{P}^d_{\mathbb{Z}}$. We can further assume that the image of $\mathcal{W}_{\mathbb{Q}}$ is exactly the rational point $W_0 = (0, \cdots, 0, 1)$ of $\mathbb{P}^d_{\mathbb{Z}}$. Denote by $\mathcal{W}_0$ the Zariski closure of W_0 in $\mathbb{P}^d_{\mathbb{Z}}$. Take the metrized line bundle $\overline{\mathcal{H}}_0 = (\mathcal{O}(1), \|\cdot\|_0)$ on $\mathbb{P}^d_{\mathbb{Z}}$ satisfying the dynamical property that the pull-back of $\overline{\mathcal{H}}_0$ by the square map is isometric to $2\overline{\mathcal{H}}_0$. Note that the Moriwaki condition $\overline{\mathcal{H}}_0^{d+1} = 0$ is satisfied. By the coordinate sections of $\mathcal{O}(1)$, we see that $\overline{\mathcal{H}}_0^d$ is represented by the arithmetic 1-cycle $(\mathcal{W}_0, \mathfrak{g}_0)$ for some positive current $\mathfrak{g}_0$ on $\mathbb{P}^d(\mathbb{C})$. Then we have

$$0 = \overline{Q} \cdot \psi^*\overline{\mathcal{H}}_0^d = \overline{Q} \cdot \psi^*(\mathcal{W}_0, \mathfrak{g}_0) \geq \overline{Q} \cdot \psi^*\mathcal{W}_0 \geq \overline{Q} \cdot \mathcal{W} \geq 0.$$

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It follows that $\overline{Q} \cdot \mathcal{W} = 0$. Here we used the nefness of $\overline{Q}$, and the inequalities can be justified by approximating $\overline{Q}$ by nef Hermitian line bundles on projective models.

Step 2. The "generic fiber" Q of $\overline{Q}$ is numerically trivial in $\widehat{\mathrm{Pic}}(\mathcal{V}_{\mathbb{Q}}/\mathbb{Q})$. In other words,

$$Q \cdot A_1 \cdots A_{d-1} = 0$$

for any $A_1, \cdots, A_{d-1} \in \mathrm{Pic}(\mathcal{B}_{m,\mathbb{Q}})$. Here $\widehat{\mathrm{Pic}}(\mathcal{V}_{\mathbb{Q}}/\mathbb{Q})$ is defined as the completion of the projective limit of Picard groups of compactifications of $\mathcal{V}_{\mathbb{Q}}$ over $\mathbb{Q}$ in the geometric setting.

By definition, we can assume that $\overline{Q}$ is the limit of a sequence of nef Hermitian line bundles $\overline{\mathcal{Q}}_m$ on projective models $\mathcal{B}_m$ of $\mathcal{V}$. We can further assume that $\mathcal{B}_m$ dominates a projective model $\mathcal{B}$ of $\mathcal{V}$, and there is an effective arithmetic divisor $\overline{\mathcal{D}} = (\mathcal{D}, g_{\mathcal{D}})$ whose finite part is supported on $\mathcal{B} \setminus \mathcal{V}$, such that

$$-\epsilon_m \overline{\mathcal{D}} < \overline{\mathcal{Q}}_m - \overline{Q} < \epsilon_m \overline{\mathcal{D}}$$

from some sequence $\epsilon_m \to 0$. Here the inequality is understood in terms of effectivity of divisors.

It follows that the height function associated to $\epsilon_m \overline{\mathcal{D}} - \overline{\mathcal{Q}}_m$ is positive on $\mathcal{V}(\overline{\mathbb{Q}})$. In particular, the height function is bounded below on any complete curve in $\mathcal{B}_{m,\mathbb{Q}}$ which intersects $\mathcal{V}_{\mathbb{Q}}$. Then the generic fiber $\epsilon_m \mathcal{D}_{\mathbb{Q}} - \mathcal{Q}_{m,\mathbb{Q}}$ is nef on such curves. This implies that $\epsilon_m \mathcal{D}_{\mathbb{Q}} - \mathcal{Q}_{m,\mathbb{Q}}$ is pseudo-effective. In fact, by Bertini's theorem, it is easy to have

$$(\epsilon_m \mathcal{D}_{\mathbb{Q}} - \mathcal{Q}_{m,\mathbb{Q}}) \cdot A_1 \cdots A_{d-1} \geq 0$$

for any ample line bundles $A_1, \cdots, A_{d-1}$ on $\mathcal{B}_{m,\mathbb{Q}}$. Set $m \to \infty$ and use the nefness of $\mathcal{Q}_{m,\mathbb{Q}}$. We have

$$Q \cdot A_1 \cdots A_{d-1} = 0.$$

The result follows by taking linear combinations.

Step 3. Let $\overline{\mathcal{H}}$ be any ample Hermitian line bundle on $\mathcal{B}$. Then

$$\lim_{m \to \infty} \lambda_1(\overline{\mathcal{H}} + \overline{\mathcal{Q}}_m) = \lambda_1(\overline{\mathcal{H}}).$$

Here the essential minimum

$$\lambda_1(\overline{\mathcal{H}}) = \lambda_1(\mathcal{V}_{\mathbb{Q}}, \overline{\mathcal{H}}) = \sup_{V' \subset \mathcal{V}_{\mathbb{Q}}} \inf_{x \in V'(\overline{\mathbb{Q}})} h_{\overline{\mathcal{H}}}(x).$$

The supremum runs through all open subschemes V' of $\mathcal{V}_{\mathbb{Q}}$.

By definition, there is a generic sequence $\{x_j\}_j$ in $\mathcal{V}(\overline{\mathbb{Q}})$ such that

$$\lim_{j \to \infty} h_{\overline{\mathcal{H}}}(x_j) = \lambda_1(\overline{\mathcal{H}}).$$

Then $\{h_{\overline{\mathcal{D}}}(x_j)\}_j$ is bounded since $\{h_{\overline{\mathcal{H}}}(x_j)\}_j$ is bounded and $\mathcal{H}_{\mathbb{Q}}$ is ample. It follows that $h_{\overline{\mathcal{Q}}_m}(x_j)$ converges uniformly to $h_{\overline{Q}}(x_j) = 0$ as $m \to \infty$. Hence,

$$\lim_{m \to \infty} \lim_{j \to \infty} h_{\overline{\mathcal{H}} + \overline{\mathcal{Q}}_m}(x_j) = \lambda_1(\overline{\mathcal{H}}).$$

It gives

$$\limsup_{m \to \infty} \lambda_1(\overline{\mathcal{H}} + \overline{\mathcal{Q}}_m) \le \lambda_1(\overline{\mathcal{H}}).$$

The other direction is trivial since $\overline{\mathcal{Q}}_m$ is nef.

Step 4. By the theorem of successive minima of [Zh1],

$$\lambda_1(\overline{\mathcal{H}} + \overline{\mathcal{Q}}_m) \ge \frac{1}{(d+1)(\mathcal{H}_{\mathbb{Q}} + \mathcal{Q}_{m,\mathbb{Q}})^d}(\overline{\mathcal{H}} + \overline{\mathcal{Q}}_m)^{d+1}.$$

Set $m \to \infty$. By the previous two steps, we end up with

$$\lambda_1(\overline{\mathcal{H}}) \ge \frac{1}{(d+1)\mathcal{H}_{\mathbb{Q}}^d}(\overline{\mathcal{H}} + \overline{Q})^{d+1}.$$

Replacing $\overline{Q}$ by a positive multiple, we see that

$$(\overline{\mathcal{H}} + t\overline{Q})^{d+1}$$

is bounded for any $t > 0$. It particularly implies that

$$\overline{Q} \cdot \overline{\mathcal{H}}^d = 0.$$

Step 5. It is formal to show that $\overline{Q}$ is numerically trivial from the property that $\overline{Q} \cdot \overline{\mathcal{H}}^d = 0$ for any ample Hermitian line bundle $\overline{\mathcal{H}}$ on $\mathcal{B}$.

In fact, for any ample Hermitian line bundles $\overline{\mathcal{H}}_1, \cdots, \overline{\mathcal{H}}_d$ on $\mathcal{B}$, we have

$$\overline{Q} \cdot (t_1 \overline{\mathcal{H}}_1 + \cdots + t_d \overline{\mathcal{H}}_d)^d = 0.$$

It is true for all positive real numbers $t_1, \cdots, t_d$, which forces

$$\overline{Q} \cdot \overline{\mathcal{H}}_1 \cdots \overline{\mathcal{H}}_d = 0.$$

By linear combinations, it is true for any Hermitian line bundles $\overline{\mathcal{H}}_1, \cdots, \overline{\mathcal{H}}_d$ on $\mathcal{B}$. By varying $\mathcal{B}$ and taking limits, it is true for any $\overline{\mathcal{H}}_1, \cdots, \overline{\mathcal{H}}_d$ in $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$.

Local version

In the end, we prove Theorem 1.5. It can be viewed as a local version of Theorem 4.9. For that purpose, we first extend the notion of f-admissibility to the local setting.

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{C}_p$. Let (X, f) be a polarizable dynamical system over $\mathbb{K}$. Assume that X is normal of dimension n. The exact sequence

$$0 \longrightarrow \operatorname{Pic}^0(X)_{\mathbb{Q}} \longrightarrow \operatorname{Pic}(X)_{\mathbb{Q}} \longrightarrow \operatorname{NS}(X)_{\mathbb{Q}} \longrightarrow 0.$$

still has a natural splitting

$$\ell_f : \operatorname{NS}(X)_{\mathbb{Q}} \longrightarrow \operatorname{Pic}(X)_{\mathbb{Q}}.$$

In fact, since $\operatorname{NS}(X)$ is a finitely generated $\mathbb{Z}$-module, we can find a finitely generated subfield K of $\mathbb{K}$ such that (X, f) and all elements of $\operatorname{NS}(X)$ are defined over K. Then the lifting ℓ_f is defined, and does not depend on the choice of K. We say that elements of $\operatorname{Pic}(X)_{\mathbb{Q}}$ in the image of ℓ_f are f-*pure of weight* 2.

Denote by $\widehat{\operatorname{Pic}}(X)$ the group of line bundles L on X, with a continuous $\mathbb{K}$-metric on the corresponding Berkovich space X^{Ber}. Note that if $\mathbb{K} = \mathbb{C}$, it is the usual complex analytic space. As in the finitely generated case, we have a unique section

$$\widehat{\ell}_{f,\mathbb{K}} : \operatorname{NS}(X)_{\mathbb{Q}} \longrightarrow \widehat{\operatorname{Pic}}(X)_{\mathbb{Q}}/\mathbb{R}^*$$

extending ℓ_f. The group $\mathbb{R}^\times$ acts on $\widehat{\operatorname{Pic}}(X)$ by scalar multiplication on the metrics.

For any $M \in \operatorname{Pic}(X)_{\mathbb{R}}$ which is f-pure of weight 2, denote by $\overline{M}_f$ the image of the algebraic equivalence class of M under $\widehat{\ell}_f$. If M is furthermore ample, then the metric of $\overline{M}_f$ is semipositive. In that case, the equilibrium measure

$$d\mu_f = \frac{1}{\deg(M)} c_1(\overline{M}_f)^n.$$

In fact, by decomposing M into f-eigencomponents. It suffices to check

$$d\mu_f = \frac{1}{M_1 \cdot M_2 \cdots M_n} c_1(\overline{M}_{1,f}) \wedge c_1(\overline{M}_{2,f}) \wedge \cdots \wedge c_1(\overline{M}_{n,f})$$

for eigenvectors $M_1, \cdots, M_n$ of f^* in $\operatorname{Pic}(X)_{\mathbb{C}}$. The identity is understood in terms of linear functionals on the space of complex-value continuous functions

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on X^{Ber}. It holds since both sides are f^*-invariant, and the uniqueness of $d\mu_f$ coming from Tate's limiting method. The following theorem refines Theorem 1.5.

Theorem 4.11. *Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{C}_p$ for some prime p. Let X be a normal projective variety over $\mathbb{K}$, and let $f, g \in \mathcal{DS}(X)$ be two polarizable algebraic dynamical system over X such that $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense in X. Then $\widehat{\ell}_{f,\mathbb{K}} = \widehat{\ell}_{g,\mathbb{K}}$ as maps from $\mathrm{NS}(X)_{\mathbb{Q}}$ to $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}/\mathbb{R}^{\times}$.*

Let us see how to obtain the result from Theorem 4.9. Let K be a finitely generated subfield of $\mathbb{K}$ such that (X, f, g) and all elements of $\mathrm{NS}(X)$ are defined over K.

Consider the inclusion $\eta : K \hookrightarrow \mathbb{K}$. By the canonical absolute value on $\mathbb{K}$, it induces a point η^{an} of $(\mathrm{Spec}\, K)^{\mathrm{an}}$. By definition, the fiber $X^{\mathrm{an}}_{\eta^{\mathrm{an}}}$ of X^{an} above η^{an} is isomorphic to X^{Ber}. For any $\xi \in \mathrm{NS}(X)_{\mathbb{Q}}$, by Theorem 4.9, we have $\widehat{\ell}_f(\xi) = \widehat{\ell}_g(\xi)$ in $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ in the setting of finitely generated fields. The identity is viewed as an equality of metrics on X^{an}. Restricted to the fiber X^{Ber}, we have $\widehat{\ell}_{f,\mathbb{K}}(\xi) = \widehat{\ell}_{g,\mathbb{K}}(\xi)$. The result is proved.

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